UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Random Matrices

Summer term 2018

Assignment 11

Due: Tuesday, July 10, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 36 (20 points). (i) Let X_N be the $N \times p$ matrix given by

$$X_N = \frac{1}{\sqrt{p}} \left(X_{ij}\right)_{\substack{1\le i\le N \\ 1\le j\le p}}.$$

Produce histograms for the eigenvalues of $X_N X_N^T$ for $N = 50$, $p = 100$ as well as for $N = 500$, $p = 1000$, where the X_{ij} are independently identically distributed ...

(1) ... Standard Gaussian random variables.

(2) ... Rademacher random variables, i.e.,

$$\mathbb{P}\left[X_{ij} = -1\right] = \frac{1}{2} = \mathbb{P}\left[X_{ij} = 1\right].$$

On the same figures, plot for $c = 0.5 = N/p$, the density of the Marchenko-Pastur distribution given by

$$\nu_c(x) = \frac{\sqrt{(\lambda^+ - x)(x - \lambda^-)}}{2\pi c x} 1_{[\lambda^-,\lambda^+]}(x),$$

where

$$\lambda^{\pm} = \left(1 \pm \sqrt{c}\right)^2$$

(ii) Let

$$X_N = \frac{1}{\sqrt{p}} \left(X_{ij}\right)_{\substack{1\le i\le N \\ 1\le j\le p}}$$

with independently identically distributed Standard Gaussian entries and consider a deterministic hermitian positive semidefinite matrix $R_N \in M_N(\mathbb{R})$.

(1) Suppose N is fixed. What is the almost sure limit of

$$R_N^{\frac{1}{2}} X_N X_N^T R_N^{\frac{1}{2}}$$

as $p \to \infty$?

(2) Assume now that $N, p \to \infty$ in such a way that

$$c_{N,p} = \frac{N}{p} \to c \in (0, \infty).$$

Let

$$R_N = \mathrm{diag}(1, \ldots, 1, 3, \ldots, 3, 7, \ldots, 7),$$

where each value appears between $\lfloor \frac{N}{3} \rfloor$ and $\lceil \frac{N}{3} \rceil$ times. Fix $N = 500$ and produce histograms for the eigenvalues of

$$R_N^{\frac{1}{2}} X_N X_N^T R_N^{\frac{1}{2}}$$

for $c = 0.1$, $c = 0.3$ and $c = 0.6$.

(3) Comment.

(iii) Suppose that $R_N = I_N + \theta_1 u_1 u_1^T + \theta_2 u_2 u_2^T$, where u_1, u_2 are deterministic vectors of norm 1. Produce histograms for the eigenvalues of

$$R_N^{\frac{1}{2}} X_N X_N^T R_N^{\frac{1}{2}}$$

for the following cases:

(1) $N = 800$, $p = 2000$, $\theta_1 = 0.1$, $\theta_2 = 0.0$

(2) $N = 800$, $p = 2000$, $\theta_1 = 1.5$, $\theta_2 = 0.0$

(3) $N = 800$, $p = 2000$, $\theta_1 = 3.0$, $\theta_2 = 3.5$

Problem 37 (15 points)**.**

Let X_N be the $N \times p$ matrix given by

$$X_N = \frac{1}{\sqrt{p}} (X_{ij})_{\substack{1 \le i \le N \\ 1 \le j \le p}}.$$

(i) Prove that $X_N X_N^*$ is positive semidefinite.

(ii) Prove that for any $z \in \mathbb{C}_+$,

$$g_{X_N^* X_N}(z) = \frac{N}{p} g_{X_N X_N^*}(z) - \frac{p - N}{nz}.$$

Suppose that the empirical spectral measure $\mu_{X_N X_N^*}$ converges weakly to a probability measure μ as $N, p \to \infty$ and $\frac{N}{p} \to c \in (0, 1)$.

(iii) What is the limit of $\mu_{X_N^* X_N}$?

Set $n = N + p$ and let A_N be the $n \times n$ Hermitian matrix given by

$$A_n = \begin{pmatrix} 0 & X_N^* \\ X_N & 0 \end{pmatrix}.$$

(iv) Prove that for any $z \in \mathbb{C}_+$,

$$z g_{A_n^2}\left(z^2\right) = g_{A_n}(z).$$

(v) Deduce that for any $z \in \mathbb{C}_+$,

$$g_{X_N X_N^*}(z) = \frac{n}{2Nz^{\frac{1}{2}}} g_{A_n}\left(z^{\frac{1}{2}}\right) + \frac{p-N}{2Nz}.$$

Problem 38 (15 points)**.** Consider the matrices $A_N, B_N \in M_N(\mathbb{R})$ given by

$$A_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ K_N & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

(i) Compute $\mathrm{rank}(A_N - B_N)$ and $\|A_N - B_N\|$.

(ii) Assume that $\sqrt[N]{K_N} \to 1$. What are the limits of μ_{A_N} and μ_{B_N}?

(iii) Find the limiting measures of $\mu_{A_N A_N^*}$ and $\mu_{B_N B_N^*}$ for any choice of K_N.

(iv) Comment.