UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Random Matrices
Winter term 2019/20

Assignment 2
Hand in on Thursday, 31.10.19, Mailbox 040.

Exercise 1 (10 points)**.**
Using your favorite programing language or computer algebra system, generate $N \times N$ random matrices for $N = 3, 9, 100$. Produce in each case a plot of the eigenvalue distribution for a single random matrix and as well a plot for the average over a high number of matrices of the considered size. The entries should be independent identically distributed (i.i.d.) according to

(i) the Bernoulli distribution $\frac{1}{2}(\delta_{-1} + \delta_1)$, where δ_x denotes the Dirac measure with atom x.

(ii) the normal distribution.

Exercise 2 (10 points)**.**
Prove Proposition 2.2 from the lecture notes, i.e. compute the moments of a standard Gaussian random variable:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n e^{-\frac{t^2}{2}} \mathrm{d}t = \begin{cases} 0 & n \text{ odd}, \\ (n-1)!! & n \text{ even}. \end{cases}$$

Exercise 3 (10 points)**.**
Let $Z_1, Z_2, \ldots, Z_n$ be independent standard complex Gaussian random variables with mean 0 and $\mathbb{E}[|Z_i|^2] = 1$ for $i = 1, \ldots, n$.

(i) Show that

$$\mathbb{E}[Z_{i_1} \cdots Z_{i_r} \overline{Z_{j_1}} \cdots \overline{Z_{j_r}}] = \#\{\sigma \in S_r \colon i_k = j_{\sigma(k)} \text{ for } k = 1, \ldots, r\}.$$

(ii) Show that

$$\mathbb{E}[Z_1^n \overline{Z_1}^m] = \begin{cases} 0 & m \neq n, \\ n! & m = n. \end{cases}$$

Exercise 4 (10 points).
Let $A = (a_{ij})_{i,j=1}^N$ be a Gaussian (GUE) random matrix with entries $a_{ii} = x_{ii}$ and $a_{ij} = x_{ij} + \sqrt{-1}y_{ij}$. Recall that this means that x_{ij} $(i \leq j)$ and y_{ij} $(i < j)$ are real i.i.d. Gaussian random variables, normalized such that $\mathbb{E}[|a_{ij}|^2] = \frac{1}{N}$. Consider the random vector

$$(x_{11}, \ldots, x_{NN}, x_{12}, \ldots, x_{1N}, \ldots, x_{N-1N}, y_{12}, \ldots, y_{1N}, \ldots, y_{N-1N})$$

in $\mathbb{R}^{N^2}$ and show that it has the density

$$C \exp(-N \frac{\mathrm{Tr}(A^2)}{2}) \mathrm{d}A,$$

where C is a constant and

$$\mathrm{d}A = \prod_{i=1}^{N} \mathrm{d}x_{ii} \prod_{i<y} \mathrm{d}x_{ij} \mathrm{d}y_{ij}.$$