UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Random Matrices
Winter term 2019/20

Assignment 3
Hand in on Monday, 10.11.19, Mailbox 040.

Exercise 1 (20 points).
Produce histograms for various random matrix ensembles

(i) Produce histograms for the averaged situation: average over 1000 realizations for the eigenvalue distribution of a an $N \times N$ Gaussian random matrix (or alternatively ± 1 entries) and compare this with one random realization for $N = 5, 50, 500, 1000$.

(ii) Check via histograms that Wigner's semicircle law is insensitive to the common distribution of the entries as long as those are independent; compare typical realisations for $N = 100$ and $N = 3000$ for different distributions of the entries: ± 1, Gaussian, uniform distribution on the interval $[-1, +1]$.

(iii) Check what happens when we give up the constraint that the the entries are centered; take for example the uniform distribution on $[0, 2]$.

(iv) Check whether the semicircle law is sensitive to what happens on the diagonal of the matrix. Choose one distribution (e.g. Gaussian) for the off-diagonal elements and another distribution for the elements on the diagonal (extreme case: put the diagonal equal to zero).

(v) Try to see what happens when we take a distribution for the entries which does not have finite second moment; for example, the Cauchy distribution.

Exercise 2 (10 points).
In class we have seen that the m-th moment of a Wigner matrix is asymptotically counted by the number of partitions $\sigma \in \mathcal{P}(m)$, for which the corresponding graph $\mathcal{G}_\sigma$ is a tree; then the corresponding walk $i_1 \to i_2 \to \cdots \to i_m \to i_1$ (where $\ker i = \sigma$) uses each edge exactly twice, in opposite directions. Assign to such a σ a pairing by opening/closing a pair when an edge is used for the first/second time in the corresponding walk.

(i) Show that this map gives a bijection between the $\sigma \in \mathcal{P}(m)$ for which $\mathcal{G}_\sigma$ is a tree and non-crossing pairings $\pi \in NC_2(m)$.

(ii) Is there a relation between σ and $\gamma\pi$, under this bijection?

Exercise 3 (10 points).
For a probability measure μ on $\mathbb{R}$ we define its Cauchy transform G_μ by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

for all $z \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. Show the following for a Cauchy transform $G = G_\mu$.

(i) $G : \mathbb{C}^+ \to C^-$, where $\mathbb{C}^- := \{z \in \mathbb{C} \mid \text{Im}(z) < 0\}$.

(ii) G is analytic on $\mathbb{C}^+$.

(iii) We have

$$\lim_{y\to\infty} iyG(iy) = 1 \qquad \text{and} \qquad \sup_{y>0, x\in\mathbb{R}} y|G(x+iy)| = 1.$$