UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Random Matrices
Winter term 2019/20

Assignment 4
Hand in on Monday, 18.11.19, Mailbox 040.

Exercise 1 (10 points).

(i) Let ν be the Cauchy distribution, i.e.,

$$d\nu(t) = \frac{1}{\pi}\frac{1}{1+t^2}dt.$$

Show that the Stieltjes transform of ν is given by

$$S(z) = \frac{1}{-i-z} \qquad \text{for } z \in \mathbb{C}^+.$$

(Note that this formula is not valid in $\mathbb{C}^-$.)
Recover from this the Cauchy distribution via the Stieltjes inversion formula.

(ii) Let A be a selfadjoint matrix in $M_N(\mathbb{C})$ and consider its spectral distribution $\mu_A = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$, where $\lambda_1, \dots, \lambda_N$ are the eigenvalues (counted with multiplicity) of A. Prove that for any $z \in \mathbb{C}^+$ the Stieltjes transform S_{μ_A} of μ_A is given by

$$S_{\mu_A}(z) = \operatorname{tr}[(A - zI)^{-1}].$$

Exercise 2 (10 points).
Let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ which converges vaguely to μ. Assume that μ is also a probablity measure. Show the following.

(i) The sequence $(\mu_N)_{N\in\mathbb{N}}$ is tight, i.e., for each $\varepsilon > 0$ there is a compact interval $I = [-R, R]$ such that $\mu_N(\mathbb{R}\backslash I) \leq \varepsilon$ for all $N \in \mathbb{N}$.

(ii) The convergence of μ_N to μ is also weakly.

Exercise 3 (10 points).
The problems with being determined by moments and whether convergence in moments implies weak convergence are mainly coming from the behaviour of our probability measures around infinity. If we restrict everything to a compact interval, then the main statements follow quite easily by relying on the Weierstrass theorem for approximating continuous functions by polynomials. In the following you should not use the general Theorem 4.14 from class.
In the following let $I = [-R, R]$ be a fixed compact interval in $\mathbb{R}$.

(i) Assume that μ is a probability measure on $\mathbb{R}$ which has its support in I (i.e., $\mu(I) = 1$). Show that all moments of μ are finite and that μ is determined by its moments (among all probability measures on $\mathbb{R}$).

(ii) Consider in addition a sequence of probability measures μ_N, such that $\mu_N(I) = 1$ for all N. Show that the following are equivalent:

- μ_N converges weakly to μ
- the moments of μ_N converge to the corresponding moments of μ