UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Random Matrices
Winter term 2019/20

Assignment 5
Hand in on Monday, 25.11.19, Mailbox 040.

In this Assignment we want to investigate the behaviour of the limiting eigenvalue distribution of matrices under certain perturbations. In order to do so, it is crucial to deal with different kinds of matrix norms. We recall the most important ones for the following exercises. Let $A \in M_N(\mathbb{C})$, then we define the following norms.

- The spectral norm (or operator norm):

$$\|A\| = \max\{\sqrt{\lambda} \colon \lambda \text{ is an eigenvalue of } AA^*\}.$$

  Some of its important properties are:

  (i) It is submultiplicative, i.e. for $A, B \in M_N(\mathbb{C})$ one has

$$\|AB\| \leq \|A\| \cdot \|B\|.$$

  (ii) It is also given as the operator norm

$$\|A\| = \sup_{\substack{x \in \mathbb{C}^N \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2},$$

  where $\|x\|_2$ is here the Euclidean 2-norm of the vector $x \in \mathbb{C}^N$.

- The Frobenius (or Hilbert-Schmidt or L^2) norm

$$\|A\|_2 = \big(\mathrm{Tr}(A^*A)\big)^{1/2} = \sqrt{\sum_{1 \leq i,j \leq N} |a_{ij}|^2}$$

Exercise 1 (10 points)**.**
In this exercise we will prove some useful facts about these norms, which you will have to use in the next exercise when adressing the problem of perturbed random matrices.
Prove the following properties of the matrix norms.

(i) For $A, B \in M_N(\mathbb{C})$ we have $|\mathrm{Tr}(AB)| \leq \|A\|_2 \cdot \|B\|_2$.

(ii) Let $A \in M_N(\mathbb{C})$ be positive and $B \in M_N(\mathbb{C})$ arbitrary. Prove that

$$|\mathrm{Tr}(AB)| \leq \|B\| \, \mathrm{Tr}(A).$$

($A \in M_N(\mathbb{C})$ is positive if there is a matrix $C \in M_N(\mathbb{C})$ such that $A = C^*C$; this is equivalent to the fact all the eigenvalues of A are real and positive.)

(iii) Let $A \in M_N(\mathbb{C})$ be normal, i.e. $AA^* = A^*A$, and $B \in M_N(\mathbb{C})$ arbitrary. Prove that

$$\max\{\|AB\|_2, \|BA\|_2\} \leq \|B\|_2 \cdot \|A\|$$

Hint: normal matrices are unitarily diagonalizable.

Exercise 2 (20 points).
In this main exercise we want to investigate the behaviour of the eigenvalue distribution of selfadjoint random matrices with respect to certain types of perturbations.

(i) Let $A \in M_N(\mathbb{C})$ be selfadjoint, $z \in \mathbb{C}^+$ and $R_A(z) = (A - zI)^{-1}$. Prove that

$$\|R_A(z)\| \leq \frac{1}{\operatorname{Im}(z)}$$

and that $R_A(z)$ is normal.

(ii) First we study a general perturbation by a selfadjoint matrix.

Let, for any $N \in \mathbb{N}$, $X_N = (X_{ij})_{i,j=1}^N$ and $Y_N = (Y_{ij})_{i,j=1}^N$ be selfadjoint matrices in $M_N(\mathbb{C})$ and define $\tilde{X}_N = X_N + Y_N$. Show that

$$\left| \operatorname{tr}(R_{\frac{1}{\sqrt{N}}X_N}(z)) - \operatorname{tr}(R_{\frac{1}{\sqrt{N}}\tilde{X}_N})(z) \right| \leq \frac{1}{(\operatorname{Im}(z))^2} \sqrt{\frac{\operatorname{tr}(Y_N^2)}{N}}$$

(iii) In this part we want to show that the diagonal of a matrix does not contribute to the eigenvalue distribution in the large N limit, if it is not too ill-behaved.

As before, consider a selfadjoint matrix $X_N = (X_{ij})_{i,j=1}^N \in M_N(\mathbb{C})$; let $X_N^D = \operatorname{diag}(X_{11}, \ldots, X_{NN})$ be the diagonal part of X_N and $X_N^{(0)} = X_N - X_N^D$ the part of X_N with zero diagonal. Assume that $\|X_N^D\|_2 \leq N$ for all $N \in \mathbb{N}$. Show that

$$\left| \operatorname{tr}(R_{\frac{1}{\sqrt{N}}X_N}(z)) - \operatorname{tr}(R_{\frac{1}{\sqrt{N}}X_N^{(0)}})(z) \right| \to 0, \quad \text{as } N \to \infty.$$