UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Random Matrices
Winter term 2019/20

Assignment 6
Hand in on Monday, 02.12.19, Mailbox 040.

Exercise 1.
We will address here concentration estimates for the law of large numbers, and see that control of higher moments allows stronger estimates. Let X_i be a sequence of independent and identically distributed random variables with common mean $\mu = E[X_i]$. We put

$$S_n := \frac{1}{n} \sum_{i=1}^{n} X_i.$$

(i) Assume that the variance $Var[X_i]$ is finite. Prove that we have then the weak law of large numbers, i.e., convergence in probability of S_n to the mean: for any $\varepsilon > 0$

$$P(\omega \mid |S_n(\omega) - \mu| \geq \varepsilon) \to 0, \qquad \text{for } n \to \infty.$$

(ii) Assume that the fourth moment of the X_i is finite, $E[X_i^4] < \infty$. Show that we have then the strong law of large numbers, i.e.,

$$S_n \to \mu, \qquad \text{almost surely.}$$

(Recall that by Borel-Cantelli it suffices for the almost sure convergence to show that

$$\sum_{n=1}^{\infty} P(\omega \mid |S_n(\omega) - \mu| \geq \varepsilon) < \infty.)$$

One should also note that our assumptions for the weak and strong law of large numbers are far from optimal. Even the existence of the variance is not needed for them, but for proofs of such general versions one needs other tools then our simple consequences of Cheyshev inequality.

Exercise 2.
Let $X_N = \frac{1}{\sqrt{N}}(x_{ij})_{i,j=1}^N$, where the x_{ij} are all (without symmetry condition) independent and identically distributed with standard complex Gaussian distribution. We denote the adjoint (i.e., congugate transpose) of X_N by X_N^*.

(i) By following the ideas from our proof of Wigner's semicircle law for the GUE in Chapter 2 show the following: the averaged trace of any $*$-moment in X_N and X_N^*, i.e.,

$$E[\text{tr}(X_N^{p(1)} \cdots X_N^{p(m)})] \qquad \text{where } p(1), \ldots, p(m) \in \{1, *\}$$

is for $N \to \infty$ given by the number of non-crossing pairings π in $NC_2(m)$ which satisfy the additional requirement that each block of π connects an X with an X^*.

(ii) Use the result from part (i) to show that the asymptotic averaged eigenvalue distribution of $W_N := X_N X_N^*$ is the same as the square of the semicircle distribution, i.e. the distribution of Y^2 if Y has a semicircular distribution.

(iii) Calculate the explicit form of the asymptotic averaged eigenvalue distribution of W_N.

(iv) Again, the convergence is here also in probability or almost surely. Produce histograms of samples of the random matrix W_N for large N and compare it with the analytic result from (iii).

Exercise 3.
We consider now random matrices $W_N = X_N X_N^*$ as before, but now we allow the X_N to be rectangular matrices, i.e., of the form

$$X_N = \frac{1}{\sqrt{p}}(x_{ij})_{\substack{1 \le i \le N \\ 1 \le j \le p}},$$

where again all x_{ij} are independent and identically distributed. We allow now real or complex entries. (In case the entries are real, X_N^* is of course just the transpose X_N^T.) Such matrices are called *Wishart matrices*. Note that we can now not multiply X_N and X_N^* in arbitrary order, but alternating products as in W_N make sense.

(i) What is the general relation between the eigenvalues of $X_N X_N^*$ and the eigenvalues of $X_N^* X_N$. Note that the first is an $N \times N$ matrix, whereas the second is a $p \times p$ matrix.

(ii) Produce histograms for the eigenvalues of $W_N := X_N X_N^*$ for $N = 50$, $p = 100$ as well as for $N = 500$, $p = 1000$, for different distributions of the x_{ij};

- standard real Gaussian random variables
- standard complex Gaussian random variables
- Bernoulli random variables, i.e., x_{ij} takes on values $+1$ and -1, each with probability $1/2$.

(iii) Compare your histograms with the density, for $c = 0.5 = N/p$, of the Marchenko-Pastur distribution which is given by

$$\frac{\sqrt{(\lambda^+ - x)(x - \lambda^-)}}{2\pi c x} 1_{[\lambda^-,\lambda^+]}(x), \qquad \text{where} \qquad \lambda^{\pm} := \left(1 \pm \sqrt{c}\right)^2.$$