UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Random Matrices
Winter term 2019/20

Assignment 8
Hand in on Monday, 06.01.20, Mailbox 040.

Exercise 1 (10 points).
In this exercise we define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

and want to show that they are the same polynomials we defined in class and that they satisfy the recursion relation. So, starting from the above definition show the following.

(i) For any $n \geq 1$,
$$xH_n(x) = H_{n+1}(x) + nH_{n-1}(x).$$

(ii) H_n is a monic polynomial of degree n. Furthermore, it is an even function if n is even and an odd function if n is odd.

(iii) The H_n are orthogonal with respect to the Gaussian measure $d\gamma(x) = (2\pi)^{-1/2} e^{-x^2/2} dx$. More precisely, show the following:
$$\int_{\mathbb{R}} H_n(x) H_m(x) d\gamma(x) = \delta_{nm} n!$$

Exercise 2 (20 points).
Produce histograms for the averaged eigenvalue distribution of a GUE(N) and compare this with the exact analytic density from class.

(i) Rewrite first the averaged eigenvalue density
$$p_N(\mu) = \frac{1}{N} K_N(\mu, \mu) = \frac{1}{\sqrt{2\pi}} \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{k!} H_k(\mu)^2 e^{-x^2/2}$$
for the unnormalized GUE(N) to the density $q_N(\lambda)$ for the normalized GUE(N) (with second moment normalized to 1).

(ii) Then average over sufficiently many normalized GUE(N), plot their histograms, and compare this to the analytic density $q_N(\lambda)$. Do this at least for $N = 1, 2, 3, 5, 10, 20, 50$.

(iii) Check also numerically that q_N converges, for $N \to \infty$, to the semicircle.

(iv) For comparison, also average over GOE(N) and over Wigner ensembles with non-Gaussian distribution for the entries, for some small N.

Motivation (For exercise 3).
We have seen that the eigenvalues of random matrices repel each other. This becomes even more apparent when we consider process versions of our random matrices, where the eigenvalue processes yield then non-intersecting paths. We want to check this numerically in this exercise. For this we consider process versions of the GUE(N) and GOE(N), those are called *Dyson Brownian motions*. They are defined as (for all $t \geq 0$) $A_N(t) := (a_{ij}(t))_{i,j=1}^N$, where each $a_{ij}(t)$ is a classical Brownian motion (complex or real) and they are independent, apart from the symmetry condition $a_{ij}(t) = \bar{a}_{ji}(t)$ for all $t \geq 0$ and all $i, j = 1, \ldots, N$. The eigenvalues $\lambda_1(t), \ldots, \lambda_N(t)$ of $A_N(t)$ give then N non-intersecting Brownian motions. We will approximate the Dyson Brownian motion by its discretized random walk version and plot the corresponding walks of the eigenvalues.

Exercise 3 (20 points).
Check the repulsion of Eigenvalues numerically by following the three steps:

(i) Approximate the Dyson Brownian motion by its discretized random walk version

$$A_N(k) := \sum_{i=1}^{k} \Delta \cdot A_N^{(i)}, \qquad \text{for } 1 \leq k \leq K$$

where $A_N^{(1)}, \ldots, A_N^{(K)}$ are K independent normalized GUE(N) random matrices. Δ is a time increment. Generate a random realization of such a Dyson random walk $A_N(k)$ and plot the N eigenvalues $\lambda_1(k), \ldots, \lambda_N(k)$ of $A_N(k)$ versus k in the same plot to see the time evolution of the N eigenvalues. Produce at least plots for three different values of N.
Hint: Start with $N = 15$, $\Delta = 0.01$, $K = 1500$, but also play around with those parameters.

(ii) For the same parameters as in part (i) consider the situation where you replace GUE by GOE and produce corresponding plots. What is the effect of this on the behaviour of the eigenvalues?

(iii) For the three considered cases of N in parts (i) and (ii), plot also N independent random walks in one plot, i.e.,

$$\tilde{\lambda}_N(k) := \sum_{i=1}^{k} \Delta \cdot x^{(i)}, \qquad \text{for } 1 \leq k \leq K$$

where $x^{(1)}, \ldots, x^{(K)}$ are K independent real standard Gaussian random variables.