10 Proof of Harer-Zagier

Let us denote

$$T(k,N) := E[\operatorname{tr}(A_N^{2k})] = \sum_{g\geq 0} \varepsilon_g(k) N^{-2g}$$

The genus expansion shows that $T(k,N)$ is, for fixed k, a polynomial in N^{-1}. Expressing it in terms of integrating over eigenvalues reveals the surprising fact that up to a Gaussian factor, it is also a polynomial in k for fixed N.

10.1 Lemma: The expression

$$N^k \frac{1}{(2k-1)!!} T(k,N)$$

is a polynomial of degree $N-1$ in k

<u>Proof</u>: Check first the easy case $N = 1$:

$T(k,1) = (2k-1)!!$ is then the $2k$-th moment of a normal variable and

$$\frac{T(k,1)}{(2k-1)!!} = 1 \text{ is a polynomial of degree } 0$$

in k.

For general N we have

$$T(k,N) = E[\operatorname{tr}(A_N^{2k})]$$

$$= c_N \cdot \int_{\mathbb{R}^N} (\lambda_1^{2k} + \dots + \lambda_N^{2k})\, e^{-\frac{N}{2}(\lambda_1^2 + \dots + \lambda_N^2)} \cdot$$

$$\cdot \underbrace{\prod_{i<j} (\lambda_i - \lambda_j)^2}_{= \prod_{i \neq j} |\lambda_i - \lambda_j|} \, d\lambda_1 \dots d\lambda_N$$

(see 7.6., we use here unordered eigenvalues)

$$= N \cdot c_N \cdot \int_{\mathbb{R}^N} \lambda_1^{2k}\, e^{-\frac{N}{2}(\lambda_1^2 + \dots + \lambda_N^2)} \cdot \prod_{i \neq j} |\lambda_i - \lambda_j|$$

$$d\lambda_1 \dots d\lambda_N$$

$$= N \cdot c_N \int_{\mathbb{R}} \lambda_1^{2k}\, p_N(\lambda_1)\, e^{-\frac{N}{2}\lambda_1^2}\, d\lambda_1$$

result of $\underbrace{\text{integrating}}$ the Vandermonde over $\lambda_2, \dots, \lambda_N$

$p_N(\lambda_1)$ is an even polynomial in λ_1, of degree $2(N-1)$; coefficients depend only on N, not on k so $\quad p_N(\lambda_1) = \sum_{\ell=0}^{N-1} d_\ell \lambda_1^{2\ell}$

$\qquad d_\ell$ can depend on N

and we get

$$T(k,N) = N \cdot c_N \sum_{\ell=0}^{N-1} d_\ell \underbrace{\int_{\mathbb{R}} \lambda_1^{2k+2\ell} e^{-\frac{N}{2}\lambda_1^2} d\lambda_1}_{k_N \cdot (2k+2\ell-1)!! \cdot N^{-k}}$$

$(2k+2\ell)$-th moment of Gauss of variance N^{-1}

$\Rightarrow N^k \dfrac{T(k,N)}{(2k-1)!!}$ is linear combination

(with N-dependent coefficients) of terms of the form

$$\frac{(2k+2\ell-1)!!}{(2k-1)!!} \quad \leftarrow \text{polynomial in } k \text{ of degree } \ell$$

This gives the assertion. $\square$

We have now that

$$N^k \frac{T(k,N)}{(2k-1)!!} = N^k \frac{1}{(2k-1)!!} \sum_{\pi \in \mathcal{P}_2(2k)} N^{\#(\gamma\pi)-k-1}$$

$$= \frac{1}{N} \underbrace{\frac{1}{(2k-1)!!} \sum_{\pi \in \mathcal{P}_2(2k)} N^{\#(\gamma\pi)}}_{=:\, t(k,N)}$$

By 10.1., $\frac{t(k,N)}{(2k-1)!!}$ is a polynomial in k of degree $N-1$; we interpret it as follows

$$t(k,N) = \sum_{\pi \in \mathcal{P}_2(2k)} \#\{\text{coloring cycles of } \gamma\pi \text{ with } \underbrace{N}_{\text{at most}} \text{ different colors}\}$$

Let us introduce

$$\tilde{t}(k,L) = \sum_{\pi \in \mathcal{P}_2(2k)} \#\{\text{coloring cycles of } \gamma\pi \text{ with } \underline{\text{exactly}}\ L \text{ different colors}\}$$

Then we have

$$t(k,N) = \sum_{L=1}^{N} \binom{N}{L} \tilde{t}(k,L)$$

number of choices of the L appearing colors among the N colors

This relation can be inverted

$$\tilde{t}(k,N) = \sum_{L=1}^{N} (-1)^{N-L} \binom{N}{L} t(k,L)$$

and hence $\frac{\tilde{t}(k,N)}{(2k-1)!!}$ is also a polynomial in k of degree $N-1$. But now we have obviously

$$0 = \tilde{t}(0,N) = \tilde{t}(1,N) = \dots = \tilde{t}(N-2,N)$$

since $\gamma\pi$ has for $\pi \in P_2(2k)$ at most $k+1$ cycles (see 2.18), and thus $\tilde{t}(k,N) = 0$ if $k+1 < N$ as we need at least N cycles if we want to use N different colors.

So, $\frac{\tilde{t}(k,N)}{(2k-1)!!}$ is in k a polynomial of degree $N-1$ and we know $N-1$ zeros, thus it must be of the form

$$\frac{\tilde{t}(k,N)}{(2k-1)!!} = \alpha_N \cdot k(k-1)(k-2)\dots(k-N+2)$$

$$= \alpha_N \binom{k}{N-1} \cdot (N-1)!$$

This gives for $t(k,N)$ the form

$$t(k,N) = \sum_{L=1}^{N} \binom{N}{L} \binom{k}{L-1} (L-1)!\, \alpha_L\, (2k-1)!!$$

To identify α_N we look at

$$\tilde{t}(N, N+1) = C_N \cdot (N+1)!$$

$$= (2N-1)!! \cdot \alpha_{N+1} \cdot \binom{N}{N} \cdot N!$$

[only the NC pairings can be colored with exactly $N+1$ colors, and for each π there are $(N+1)!$ ways of coloring]

$$\Rightarrow \alpha_{N+1} = \frac{C'_N \cdot (N+1)}{(2N-1)!!}$$

$$= \frac{1}{(2N-1)!!} \binom{2N}{N}$$

$$= \frac{2^N}{N!}$$

thus we have

$$T(k,N) = \frac{1}{N^{k+1}} \, t(k,N)$$

$$= \frac{1}{N^{k+1}} \sum_{L=1}^{N} \binom{N}{L} \binom{k}{L-1} \cancel{(L-1)!} \frac{2^{L-1}}{\cancel{(L-1)!}} (2k-1$$

$$= (2k-1)!! \frac{1}{N^{k+1}} \sum_{L=1}^{N} \binom{N}{L} \binom{k}{L-1} 2^{L-1}$$

To get from this information how this changes in k, we consider a generating function in k

$$T(s,N) := 1 + 2\sum_{k=0}^{\infty} \frac{T(k,N)}{(2k-1)!!} (Ns)^{k+1}$$

$$= 1 + 2\sum_{k=0}^{\infty} \sum_{L=1}^{N} \binom{N}{L} \binom{k}{L-1} 2^{L-1} s^{k+1}$$

$$= \sum_{L=0}^{N} 2^L \binom{N}{L} \underbrace{\sum_{k=L-1}^{\infty} \binom{k}{L-1} s^{k+1}}_{= \left(\frac{s}{1-s}\right)^L}$$

$$= \sum_{L=0}^{N} \binom{N}{L} \left(\frac{2s}{1-s}\right)^L$$

$$= \left(1 + \frac{2s}{1-s}\right)^N$$

$$= \left(\frac{1+s}{1-s}\right)^N$$

Note that

$$\frac{1}{(2k-1)!!} = \frac{2^k}{k!\, G_k\, (k+1)}$$ (see calculation above for α_k

and hence $T(s,N)$ can also be rewritten as generating function in our

$$b_k^{(N)} = \frac{T(k,N)}{G_k}$$ (note that $b_k = b_k^{(N)}$ depend on N)

as

$$T(s,N) = 1 + 2\sum_{k=0}^{\infty} \frac{T(k,N)}{(k+1)!\, G_k} 2^k (Ns)^{k+1}$$

$$= 1 + \sum_{k=0}^{\infty} \frac{b_k^{(N)}}{(k+1)!} (2Ns)^{k+1} \qquad (*)$$

To get a recursion for the $b_k^{(N)}$, we need some functional relation for $T(s,N)$.
Note that the recursion in Harer-Zagier involves b_k, b_{k+1}, b_{k-1} for the same N, thus we need a relation which does not change the N.

$$T(s,N) = \left(\frac{1+s}{1-s}\right)^N$$

$$\Rightarrow \frac{d}{ds} T(s,N) = N \cdot \left(\frac{1+s}{1-s}\right)^{N-1} \cdot \frac{1-s+(1+s)}{(1-s)^2}$$

$$= 2N \left(\frac{1+s}{1-s}\right)^N \frac{1}{(1-s)(1+s)}$$

$$= 2N \; T(s,N) \; \frac{1}{1-s^2}$$

and thus

$$(1-s^2) \frac{d}{ds} T(s,N) = 2N \; T(s,N)$$

If we compare coefficients of s^{k+1} for the generating series $(*)$, we get

note:

$$\frac{d}{ds} T(s,N) = \sum_{k=0}^{\infty} \frac{b_k^{(N)}}{k!} (2Ns)^k \; 2N$$

$$\frac{b_{k+1}^{(N)}}{(k+1)!}(2N)^{k+2} - \frac{b_{k-1}^{(N)}}{(k-1)!}(2N)^{k} = 2N\,\frac{b_{k}^{(N)}}{(k+1)!}(2N)^{k}$$

$$\Rightarrow\; b_{k+1}^{(N)} = b_{k}^{(N)} + b_{k-1}^{(N)} \cdot \frac{(k+1)\,k}{(2N)^2} \qquad \square$$