11. Statistics of longest increasing subsequence

11.1. Def.: A permutation $\sigma \in S_n$ is said to have an increasing subsequence of length k if there exist indices

$$1 \leq i_1 < i_2 < \dots < i_k \leq n$$

such that

$$\sigma(i_1) < \sigma(i_2) < \dots < \sigma(i_k)$$

For a decreasing subsequence of length k the above holds with the second set of inequalities reversed.

For a given $\sigma \in S_n$, we denote the length of an increasing subsequence of maximal length by $L_n(\sigma)$.

11.2. Examples: 1) $\sigma = id = \begin{matrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{matrix}$

has an increasing subsequence of length n, hence $L_n(id) = n$, and all decreasing subsequences have length 1

2) $\sigma_0 = \begin{matrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & & 2 & 1 \end{matrix}$ has $L_n(\sigma_0) = 1$

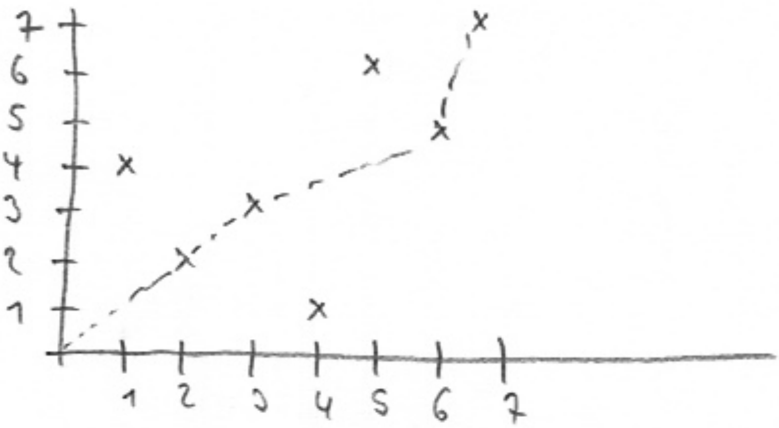
but then is an decreasing subsequence of length

3) Consider $\sigma = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & \boxed{2} & \boxed{3} & 1 & 6 & \boxed{5} & \boxed{7} \\ & \boxed{} & \boxed{} & & \boxed{} & & \boxed{} \end{array}$

$\{2,3,5,7\}$ or $\{2,3,6,7\}$ are longest increasing subsequences, thus $L_7(\sigma) = 4$

$\{4,3,1\}$ or $\{4,2,1\}$ are decreasing subsequences of length 3

In the graphical representation



a longest subsequence corresponds to a path that always goes up.

4) Longest increasing subsequences are relevant for sorting algorithms.
Consider a library of n books, labelled bijectively with numbers $\{1, \dots, n\}$, arranged somehow on a single long bookshelf.
The configuration of the books corresponds to a permutation $\sigma \in S_n$. How many operations

does one need to sort the books in canonical ascending order $1, 2, \dots, n$.
It turns out that the minimum number is $n - L_n(\sigma)$. One can sort around a increasin subsequence

Example: $4\ \boxed{1}\ 9\ 3\ \boxed{2}\ 7\ \boxed{6}\ \boxed{8}\ 5$

$4\ \boxed{1}\ 9\ \boxed{2}\ 3\ 7\ \boxed{6}\ \boxed{8}\ 5$

$1\ 9\ 2\ 3\ 4\ 7\ 6\ 8\ 5$

$1\ 9\ 2\ 3\ 4\ 5\ 7\ 6\ 8$

$1\ 9\ 2\ 3\ 4\ 5\ 6\ 7\ 8$

$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9$

$9 - 4 = 5$ operations to sort

11.3 Remark: One has situations with only small increasing subsequences, like $\sigma = n\ n-1 \dots 3\ 2\ 1$, but then one has long decreasing subsequences. This is true in general, one cannot avoid both long decreasing and increasing subsequences at the same time, according to the slogan "complete disorder is impossible" C.Motzkin

11.4. Theorem (Erdős + Szekeres, 1935):

Every permutation $\sigma \in S_{n^2+1}$ has a monotone subsequence of length more than n.

Proof: Write $\sigma = a_1 a_2 \dots a_{n^2+1}$

Assign label (x_k, y_k), where

$x_k :=$ length of longest increasing subsequence ending at a_k

$y_k :=$ — " — decreasing — " —

Assume now that there is no monotone subsequence of length $n+1$

$\Rightarrow 1 \le x_k, y_k \le n \quad \forall k$

i.e. there are only n^2 possible labels

pigeonhole principle $\Rightarrow \exists i \neq j : (x_i, y_i) = (x_j, x_j)$

take $i < j$: $a_i < a_j \rightsquigarrow$ append a_j to longest increasing subsequence ending at a_i

$\Rightarrow x_j > x_i$

$a_i > a_j \rightsquigarrow$ append a_j to longest decreasing subsequence ending at a_i $\Rightarrow y_j > y_i$

contradiction $\square$

11.5 History: We are now interested in the statistics of $L_n(\sigma)$, for $n \to \infty$. This means, we put uniform distribution on permutations, i.e. $P(\sigma) = \frac{1}{n!}$ $\forall \sigma \in S_n$ and consider $L_n : S_n \to \mathbb{R}$ as random variable. What is the asymptotic distribution of L_n? (Ulam's problem, raised in the 1960's)

Hammersley 1972: the limit

$$\Lambda = \lim_{n \to \infty} \frac{E[L_n]}{\sqrt{n}}$$

exists and

$$\frac{L_n}{\sqrt{n}} \to \Lambda \quad \text{in probability}$$

Vershik + Kerov, Logan + Shep 1977: $\Lambda = 2$

$$\text{i.e. } E[L_n] \approx 2\sqrt{n} + o(\sqrt{n})$$

Baik, Deift, Johansson 1998:

$$E[L_n] \approx 2\sqrt{n} + c\, n^{1/6} + o(n^{1/6})$$

more precisely

$$\lim_{n \to \infty} P\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \le t\right) = F_2(t) \quad \text{Tracy-Widom distribution}$$

11.6. "Sketch" of proof of BDJ theorem:

1) RSK correspondence relates permutations to Young diagrams. L_n goes under this mapping to length of first row of diagram

2) These Young diagrams correspond to non-intersecting paths

3) Via Gessel-Viennot the relevant quantities in terms of non-intersecting paths have a determinantal form

4) Show that the involved kernel, suitably rescaled, converges to the Airy kernel.

11.7. RSK - correspondence:

$$\text{RSK} = \underset{\substack{\uparrow \\ 1938}}{\text{Robinson}} - \underset{\substack{\uparrow \\ 1961}}{\text{Schensted}} - \underset{\substack{\uparrow \\ 1973}}{\text{Knuth}}$$

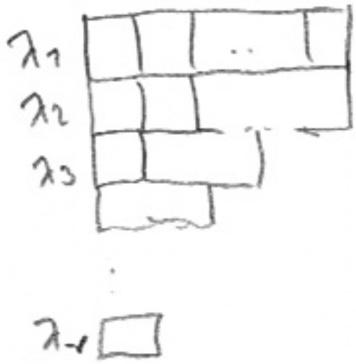
gives a bijection

$$S_n \longleftrightarrow \bigcup_{\substack{\lambda \\ \text{Young diagram} \\ \text{of size } n}} (\text{Tab}\,\lambda) \times \underset{\substack{\uparrow \\ \text{Young tableaux} \\ \text{of shape } \lambda}}{(\text{Tab}\,\lambda)}$$

11.8 Def: 1) Let $n \geq 1$. A *partition of n* is a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1$ and $\sum_i \lambda_i = n$.

We denote this by $\lambda \vdash n$.

Graphically a partition $\lambda \vdash n$ is represented by a *Young diagram* with n boxes.



2) A *Young tableau* of shape λ is the Young diagram λ filled with numbers $1, \dots, n$ such that: in any row, the numbers are increasing from left to right and in any column the numbers are increasing from top to bottom. We denote the set of Young tableaux of shape λ by $\mathrm{Tab}\,\lambda$

11.9 Examples and Remarks:

1) $n = 1$, $\lambda = \square$, $\mathrm{Tab}(\square) = \boxed{1}$

$n = 2$, $\lambda = \square\square$ or $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

$2 = 2$ $\quad$ $2 = 1+1$

$\mathrm{Tab}(\square\square) = \boxed{1\,|\,2}$

$\mathrm{Tab}\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = \begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}$

$n=3$, $\lambda = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \end{array}$, $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array}$

$3=3 \qquad 3=2+1 \qquad 3=1+1+1$

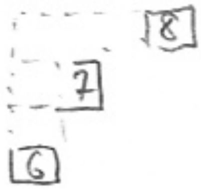
$$\mathrm{Tab}\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \end{array}\right) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}$$

2) Note that a tableaux of shape λ corresponds to a walk from ϕ to λ by adding one box in each step and only using Young diagrams.

e.g. $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$ corresponds to

$$\phi \to \begin{array}{|c|}\hline 1 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$\to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \to$$



3) Those objects are extremely important, since they parametrize the irreducible representations of S_n

$$\lambda \vdash n \longleftrightarrow \text{irreducible repr. } \pi_\lambda \text{ of } S_n$$

$$\# \mathrm{Tab}(\lambda) = \dim \pi_\lambda$$

Recall that one has for finite groups

$$\sum_{\lambda \vdash n} (\dim \pi_\lambda)^2 = \# S_n = n!$$

e.g.: $n=3$
$6 = 1+4+1$

The latter suggests that there should be a bijection

$$S_n \longleftrightarrow \bigcup_{\lambda \vdash n} (\mathrm{Tab}\,\lambda) \times (\mathrm{Tab}\,\lambda)$$

Such a bijection is given by RSK correspondence

Example

$$\sigma = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{matrix} \longleftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad (*$$

$$L_7(\sigma) = 4 \quad \longleftrightarrow \quad \text{length first row} = 4$$

11.10 Relation to non-intersecting paths:

pairs $(Q, P) \in (\mathrm{Tab}\,\lambda \times \mathrm{Tab}\,\lambda)$ can be identified with $r = \#\mathrm{rows}(\lambda)$ paths

Q gives positions of going up

P gives positions of going down

e.g. take (*) from above