12. The circular law

The non-selfadjoint analogue of the GUE is given by the Ginibre ensemble, where all entries are independent and complex Gaussians.

A standard complex Gaussian is of the form

$z = \frac{x+iy}{\sqrt{2}}$, where x, y are independent standard real Gaussians, i.e. with joint distribution

$$p(x,y)\,dx\,dy = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}\,dx\,dy$$

If we rewrite this in terms of a density with respect to Lebesgue measure for real and imaginary part of

$$z = \frac{x+iy}{\sqrt{2}} = t_1 + i t_2$$

$$\bar{z} = \frac{x-iy}{\sqrt{2}} = t_1 - i t_2$$

we get

$$p(t_1,t_2)\,dt_1\,dt_2 = \frac{1}{\pi} e^{-(t_1^2+t_2^2)}\,dt_1\,dt_2$$

$$= \frac{1}{\pi} e^{-|z|^2}\,d^2z$$

(where $d^2z \mathrel{\hat{=}} dt_1\,dt_2$)

12.1 Def: A (complex) unnormalized Ginibre ensemble

$A = (a_{ij})_{i,j=1}^{N}$ is given by complex-valued entries with joint distribution

$$\frac{1}{\pi^{N^2}} e^{-\sum_{i,j=1}^{N} |a_{ij}|^2} dA = \frac{1}{\pi^{N^2}} e^{-\operatorname{Tr}(AA^*)} dA,$$

where $dA = \prod_{i,j=1}^{N} d^2 a_{ij}$

As for the GUE case (see Theorem 7.6.) we can rewrite the density in terms of eigenvalues. Note that the eigenvalues are now complex.

12.2. Theorem: The joint distribution of the complex eigenvalues of a $N \times N$ Ginibre ensemble is given by a density

$$p(z_1, \dots, z_N) = c_N \exp\left(-\sum_{k=1}^{N} |z_k|^2\right) \prod_{1 \le i < j \le N} |z_i - z_j|^2$$

12.3. Remark: 1) Note that typically Ginibre matrices are not normal, i.e. $AA^* \neq A^*A$. This means that one looses the relation between functions in the eigenvalues and traces of functions of the matrix.

The latter is what we can control, the former is what we need in order to understand the eigenvalues.

2) As in the selfadjoint case, the eigenvalues repell, hence there will, almost surely, be no multiple eigenvalues. Thus we can also in the complex case diagonalize our matrix, i.e.

$$A = V D V^{-1}$$

where $D = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_N \end{pmatrix}$ contains the eigenvalues

However, V is now not unitary any more (i.e. eigenvectors for different eigenvalues are in general not orthogonal); we can also diagonalize A^* via

$$A^* = (V^{-1})^* D^* V^*$$

but since $V^{-1} \neq V^*$ (if A is not normal) we cannot diagonalize A and A^* simultaneously

This means that in general for example $\mathrm{Tr}(A A^* A^* A)$ has no clear relation to $\sum_{i=1}^{N} z_i \bar{z}_i \bar{z}_i z_i$

[Note that $\mathrm{Tr}(AA^*A^*A) \neq \mathrm{Tr}(AA^*AA^*)$
if $AA^* \neq A^*A$, but of course

$$\sum z_i \bar{z}_i \bar{z}_i z_i = \sum z_i \bar{z}_i z_i \bar{z}_i$$

3) In 12.2 it seems that we have rewritten the density $\exp[-\mathrm{Tr}(AA^*)]$ as $\exp(-\sum |z_k|^2)$. However, this is more subtle. One can bring any matrix via a unitary conjugation in a triangular form (Schur decomposition):

$$A = \underset{\text{unitary}}{\mathcal{U}}\, T\, \mathcal{U}^* \qquad T = \begin{pmatrix} z_1 & & * \\ & \ddots & \\ 0 & & z_N \end{pmatrix} = (t_{ij})$$

$z_1, \dots, z_N$ eigenvalues of A

Then $A^* = \mathcal{U}\, T^*\, \mathcal{U}^*$ with

$$T^* = \begin{pmatrix} \bar{z}_1 & & 0 \\ & \ddots & \\ * & & \bar{z}_N \end{pmatrix}$$ and

$$\mathrm{Tr}(AA^*) = \mathrm{Tr}(TT^*) = \sum_{j \geq i} t_{ij} \bar{t}_{ij}$$
$$= \sum |z_i|^2 + \sum_{j > i} t_{ij} \bar{t}_{ij}$$

Integrating out the t_{ij} $(j>i)$ gives density for z_i

4) As for the GUE case (Theorem 6.15) we can now write the Vandermonde density in determinantal form; the only difference is that we have to replace the Hermite polynomials $H_k(x)$, which orthogonalize the real Gauß distribution, by monomials z^k, which orthogonalize the complex Gauß distribution

<u>12.4. Theorem:</u> The joint eigenvalue distribution of the Ginibre ensemble is of the determinantal form

$$p(z_1,\dots,z_N) = \frac{1}{N!} \det \left(K_N(z_i,z_j)\right)_{i,j=1}^{N}$$

with the kernel

$$K_N(z,w) = \sum_{k=0}^{N-1} \varphi_k(z)\overline{\varphi_k(w)}$$

with

$$\varphi_k(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}|z|^2} \frac{1}{\sqrt{k!}} z^k$$

In particular, we have for the averaged eigenvalue density of an unnormalized Ginibre random matrix the density

$$p_N(z) = \frac{1}{N} K_N(z,z) = \frac{1}{N\pi} e^{-|z|^2} \sum_{k=0}^{N-1} \frac{|z|^{2k}}{k!}$$

12.5 Theorem (Circular law for Ginibre):

The averaged eigenvalue distribution for a normalized Ginibre random matrix $\frac{1}{\sqrt{N}} A_N$ converges for $N \to \infty$ weakly to the uniform distribution on the unit disc of $\mathbb{C}$ with density

$$z \mapsto \frac{1}{\pi} 1_{\{z \in \mathbb{C} : |z| \leq 1\}}$$

Proof: The density q of the normalized Ginibre is given by

$$q_N(z) = N\, p_N(\sqrt{N} z)$$

$$= \frac{1}{\pi} e^{-N|z|^2} \sum_{k=0}^{N-1} \frac{(N|z|^2)^k}{k!}$$

We have to show that this converges to the circular density

We have for $|z| < 1$:

$$e^{N|z|^2} - \sum_{k=0}^{N-1} \frac{(N|z|^2)^k}{k!} = \sum_{k=N}^{\infty} \frac{(N|z|^2)^k}{k!}$$

$$\leq \frac{(N|z|^2)^N}{N!} \cdot \sum_{\ell=0}^{\infty} \frac{(N|z|)^\ell}{(N+1)^\ell} = \frac{(N|z|^2)^N}{N!} \frac{1}{1 - \frac{N|z|^2}{N+1}}$$

$$\Rightarrow 1 - e^{-N|z|^2} \sum_{k=0}^{N-1} \frac{(N|z|^2)^k}{k!} \leq$$

$$\leq \underbrace{e^{-N|z|^2} \cdot \frac{(N|z|^2)^N}{N!}}_{\xrightarrow{N\to\infty} 0} \quad \frac{1}{1 - \frac{N|z|^2}{N+1}} \quad \xrightarrow{N\to\infty} \frac{1}{1-|z|^2}$$

and for $|z| > 1$:

$$\sum_{k=0}^{N-1} \frac{(N|z|^2)^k}{k!} \leq \frac{(N|z|^2)^{N-1}}{(N-1)!} \cdot \underbrace{\sum_{l=0}^{N-1} \frac{(N-1)^l}{(N|z|^2)^l}}_{\leq \sum_{l=0}^{\infty} \dots}$$

$$= \frac{1}{1 - \frac{N-1}{N|z|^2}}$$

$$\Rightarrow e^{-N|z|^2} \sum_{k=0}^{N-1} \frac{(N|z|^2)^k}{k!} \leq$$

$$\leq \underbrace{e^{-N|z|^2} \cdot \frac{(N|z|^2)^{N-1}}{(N-1)!}}_{\xrightarrow{N\to\infty} 0} \cdot \frac{1}{1 - \frac{N-1}{N|z|^2}} \quad \xrightarrow{N\to\infty} \frac{1}{1-\frac{1}{|z|^2}}$$

12.6. Remarks: 1) The convergence holds also almost surely.

2) The circular law holds also for non-Gaussian entries, but to prove this is much harder than the extension of the semicircle law from Gaussian to Wigner matrices.

12.7. Theorem: Consider a complex random matrix $A_N = \frac{1}{\sqrt{N}}(a_{ij})_{i,j=1}^{N}$ where the a_{ij} are independent and identically distributed complex random variables with variance 1, i.e.

$$E[|a_{ij}|^2] - E[a_{ij}]^2 = 1.$$

Then the eigenvalue distribution of A_N converges weakly almost surely for $N \to \infty$ to the uniform distribution on the unit disc.

12.8. History of Proof:

~ 60's ; Mehta : in expectation for Ginibre ensemble

~ 80's , Silverstein : almost sure convergence for Ginibre

~ 80's, 90's, Girko : ideas for proof in general case,

1997 Bai : first rigorous proof under additional assumptions on distribution

... papers by Tao, Vu; Götze, Tikhomirov; Pan, Zhou and others improving more and more on optimal assumptions

2010, Tao+Vu: final version under assumption of existence of second moment

12.9. Remarks: 1) For measures on $\mathbb{C}$ one can use *-moments to describe them, but to control
or Stieltjes transforms
convergence properties is the main problem.

2) For a matrix A its *-moments are all expressions of the form

$$tr(A^{\varepsilon(1)} \dots A^{\varepsilon(m)})$$

where $m \in \mathbb{N}$ and $\varepsilon(1), \dots, \varepsilon(m) \in \{1, *\}$

The eigenvalue distribution of A

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N}) \qquad (\lambda_1, \dots, \lambda_N \text{ eigenvalues of } A)$$

is uniquely determined by the knowledge of all *-moments of A, but convergence of *-moments does not necessarily imply convergence of the eigenvalue distribution

Example: Consider

$$A_N = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & \\ 0 & & \ddots & 1 \\ & & & 0 \end{pmatrix} \quad \text{and} \quad B_N = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix}$$

Then

$\mu_{A_N} = \delta_0$ and μ_{B_N} = uniform distribution on N-th roots of unity

hence

$\mu_{A_N} \to \delta_0$ and $\mu_{B_N} \to$ uniform distribution on circle

but the limits of the $*$-moments are the same for A_N and B_N

3) For each measure μ on $\mathbb{C}$ one has the Cauchy-Stieltjes transform

$$m_\mu(z) := \int_{\mathbb{C}} \frac{1}{\lambda - z} \, d\mu(\lambda)$$

This is almost surely defined.
However, it is analytic in z only outside the support of μ; in order to recover μ from m_μ one needs also the info about m_μ inside the support.

In order to determine and deal with μ_A one reduces it via Girko's "hermitization method"

$$\int_{\mathbb{C}} \log|\lambda - z| \, d\mu_A(z) = \int_0^\infty \log(t) \, d\mu_{|A-\lambda 1|}(t) \qquad (*)$$

to selfadjoint matrices

LHS of (*) for all λ determines μ_A

RHS is about selfadjoint matrices

$$|A - \lambda 1| = \sqrt{(A - \lambda 1)(A^* - \bar{\lambda} 1)}$$

Note that the eigenvalues of $|B|$ are related to those of

$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$$

In this analytic approach one still needs to control convergence properties. For this estimates of probabilities of small singular values are crucial.

→ Survey of Bordenave + Chafaï

"Around the circular law"