13. Several independent GUE and asymptotic freeness

Up to now we have mainly considered limits $N \to \infty$ of one random matrix A_N. But often one has several random matrix ensembles and would like to understand their "joint" distribution; e.g. in order to use them as building blocks for more complicated random matrix models.

[An example of this occurred in Problem 36, where $R_N^{1/2} X_N X_N^T R_N^{1/2}$ is built out of the two random matrices X_N and R_N – note that a deterministic matrix (like R_N) is also a special case of a random matrix.]

13.1. Remarks: 1) Consider two random matrices
$$A_N = (a_{ij})_{i,j=1}^N, \quad B_N = (b_{ij})_{i,j=1}^N$$
where the entries $\{a_{ij}\}$ and $\{b_{kl}\}$ are defined on the same probability space.

What is now the "joint" information about A_N and B_N in which we are (for $N \to \infty$) interested. Note that in general our analytic approach breaks down; if A_N and B_N do not commute (as in interesting cases), then we cannot diagonalize them simultaneously, hence it makes no sense to talk of a joint eigenvalue distribution of A_N, B_N. The notion μ_{A_N,B_N} has no clear analytic meaning.

What still makes sense in the multi-variate case is the combinatorial approach via "mixed" moments with respect to the trace tr. Hence we will consider the collection of all mixed moments in A_N, B_N,

$$tr[A_{i_1}^{(N)} \dots A_{i_m}^{(N)}] \qquad m \in \mathbb{N},\ 1 \le i_1, \dots, i_m \le 2$$

(with the notation $A_1^{(N)} := A_N$, $A_2^{(N)} := B_N$)

as the joint distribution of A_N, B_N, and denote this also by μ_{A_N,B_N}.

We want to understand in interesting cases the behaviour of μ_{A_N,B_N} for $N \to \infty$

2) In the case of one (selfadjoint) matrix A the notion μ_A has two meanings:

analytic: $\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$ is a probability measure on $\mathbb{R}$

combinatorial: μ_A is given by all moments $tr[A^m]$ for all $m \in \mathbb{N}$

These two points of view are the same via

$$\operatorname{tr}[A^m] = \int t^m \, d\mu_A(t)$$

(usually we only consider cases where the probability measure μ_A is determined by its moments)

In the case of two matrixes A_1, A_2 the notion $\mu_{A_1 A_2}$ has only one meaning, namely the collection of all moments

$$\operatorname{tr}[A_{i(1)} \cdots A_{i(m)}], \quad m \in \mathbb{N}, \; i(1), \dots, i(m) \in \{1,2\}$$

If A_1, A_2 do not commute there does not exist a probability measure μ on $\mathbb{R}^2$ with

$$\operatorname{tr}[A_{i(1)} \cdots A_{i(m)}] = \int t_{i(1)} \cdots t_{i(m)} \, d\mu(t_1, t_2)$$

for all $m \in \mathbb{N}$, $i(1), \dots, i(m) \in \{1,2\}$

(This is the same kind of problem which we addressed in 12.3. for one non-normal matrix.)

<u>13.2. Joint moments of independent GUEs:</u>

We will now consider the simplest case of several random matrices, namely r GUEs $A_1^{(N)}, \dots, A_r^{(N)}$ which we assume to be independent of each other, i.e. we have $A_i = \frac{1}{\sqrt{N}} \left(a_{kl}^{(i)}\right)_{k,l=1}^{N}$

where each A_i $(i = 1, \dots, r)$ is a GUE and $\{a_{kl}^{(1)} \mid k,l = 1, \dots, N\}, \{a_{kl}^{(2)} \mid k,l = 1, \dots, N\}, \dots, \{a_{kl}^{(r)} \mid k,l$ are independent sets of random variables

Equivalently, this can be characterized that all entries of all matrices together form a collection of independent standard Gaussian random variables (real on the diagonals, complex otherwise). Hence we can express this again in terms of the Wick formula as

$$E[a^{(i(1))}_{k(1)\ell(1)} a^{(i(2))}_{k(2)\ell(2)} \cdots a^{(i(m))}_{k(m)\ell(m)}] =$$

$$\sum_{\pi \in P_2(m)} E_\pi[a^{(i(1))}_{k(1)\ell(1)}, \dots, a^{(i(m))}_{k(m)\ell(m)}]$$

$$\forall m;\ 1 \le k(1), \ell(1), \dots, k(m), \ell(m) \le N$$
$$1 \le i(1), \dots, i(m) \le r$$

and where the second moments are given by

$$E[a^{(i)}_{pq} a^{(j)}_{k\ell}] = \delta_{p\ell}\, \delta_{qk}\, \delta_{ij}$$

We can now essentially repeat the calculation from 2.12. for our mixed moments:

$$E[tr(A_{i_1} \cdots A_{i_m})] =$$

$$= \frac{1}{N^{m/2+1}} \sum_{k_1, \dots, k_m = 1}^{N} \underbrace{E[a^{(i_1)}_{k_1k_2} a^{(i_2)}_{k_2k_3} \cdots a^{(i_m)}_{k_mk_1}]}$$

$$\underbrace{\sum_{\pi \in P_2(m)} E_\pi[a^{(i_1)}_{k_1k_2}, \dots, a^{(i_m)}_{k_mk_1}]}$$

$$\prod_{(p,q) \in \pi} \underbrace{E[a^{(i_p)}_{k_pk_{p+1}} a^{(i_q)}_{k_qk_{q+1}}]}$$

$$= \frac{1}{N^{m/2+1}} \sum_{\substack{\pi \in P_2(m) \\ i_p = i_q \\ \forall (p,q) \in \pi}} \underbrace{\sum_{k_1,\dots,k_m=1}^{N} \prod_p [k_p = k_{\gamma\pi(p)}]}_{N^{\#(\gamma\pi)}}$$

Hence we get the same kind of genus expansion for several GUE as for one GUE; the only difference is that our pairings are only allowed to connect the same matrices.

13.3. Notation: For a given $i = (i_1, \dots, i_m)$ (with $1 \le i_1, \dots, i_m \le r$) we say that $\pi \in P_2(m)$ respects i if we have $i_p = i_q$ for all pairs $(p,q) \in \pi$. We put

$$P_2^{(i)}(m) := \{\pi \in P_2(m) \mid \pi \text{ respects } i\}$$

and also

$$NC_2^{(i)}(m) := \{\pi \in NC_2(m) \mid \pi \text{ respects } i\}$$

13.4. Theorem (Genus expansion for independent GUEs): Let $A_1, \dots, A_r$ be r independent $N \times N$-GUE. Then we have for all $m \in \mathbb{N}$ and all $i_1, \dots, i_m \in [r]$

$$E[\mathrm{tr}(A_{i_1} \cdots A_{i_m})] = \sum_{\pi \in P_2^{(i)}(m)} N^{\#(\gamma\pi) - \frac{m}{2} - 1}$$

and thus

$$\lim_{N \to \infty} E[\mathrm{tr}(A_{i_1} \cdots A_{i_m})] = \# NC_2^{(i)}(m)$$

Proof: The genus expansion follows from our calculation. The limit $N \to \infty$ follows as in 2.14. by the fact that for $\pi \in P_2(m)$:

$$\lim_{N\to\infty} N^{\#(\gamma\pi)-\frac{m}{2}-1} = \begin{cases} 1 & \pi \in NC_2(m) \\ 0 & \pi \notin NC_2(m) \end{cases}$$

The index tuple $(i_1, .., i_m)$ has no say in this limit.

13.5. Remark: We would like to find some structure in those limiting moments. We prefer to talk directly about the limit instead of making asymptotic statements. In the case of one GUE, we had the semicircle μ_W as a limiting analytic object. Now we don't have an analytic object in the limit, but we can organize our our distribution as the limit of moments in a more algebraic way.

13.6. Definitions: 1) Let $\mathcal{A} = \mathbb{C}\langle s_1, .., s_r \rangle$ be the algebra of polynomials in the non-commuting variables $s_1, .., s_r$; this means $\mathcal{A}$ is the free unital algebra generated by $s_1, .., s_r$ (i.e. there are no non-trivial relations among the $s_1, .., s_r$) and $\mathcal{A}$ is spanned linearly by monomials $s_{i_1} \cdots s_{i_m}$ (for $m \geq 0$, $i_1, .., i_m \in [r]$; $m = 0$ corresponds to the unit 1) and multiplication of monomials is just concatenation.

2) On this $\mathcal{A}$ we define now a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ by

$$\varphi(1) = 1$$

$$\varphi(s_{i_1} \cdots s_{i_m}) := \lim_{N\to\infty} E[\operatorname{tr}(A_{i_1} \cdots A_{i_m})]$$

$$= \# NC_2^{(i)}(m)$$

for all $m \geq 0$; $i_1, \dots, i_m \in [r]$

3) We also address $(\mathcal{A}, \varphi)$ as a <u>non-commutative probability space</u> and $s_1, \dots, s_r \in \mathcal{A}$ as (non-commutative) <u>random variables</u>. The <u>moments</u> of $s_1, \dots, s_r$ are the $\varphi(s_{i_1} \cdots s_{i_m})$ and the collection of all those moments is the <u>(joint) distribution</u> of $s_1, \dots, s_r$

<u>13.7. Remarks:</u> 1) Note that if we consider only one of the s_i, then its distribution is just the collection of Catalan numbers, hence correspond to the semicircle, which we understand quite well

2) If we consider all $s_1, \dots, s_r$, then their joint distribution is a huge collection of numbers. We claim that the following theorem discovers some important structure in those.

<u>13.8 Theorem</u>: Let $\mathcal{A} = \mathbb{C}\langle s_1, \dots, s_r \rangle$ and

$\varphi : \mathcal{A} \to \mathbb{C}$ defined by $\varphi(s_{i_1} \cdots s_{i_m}) = \# NC_2^{[i]}(m)$

be as before. Then we have for

- all $m \geq 1$
- all $i_1, \dots, i_m \in [r]$ with
  $i_1 \neq i_2, i_2 \neq i_3, i_3 \neq i_4, \dots, i_{m-1} \neq i_m$
- all polynomials $p_1, \dots, p_m$ in one variable
  such that $\varphi(p_k(s_{i_k})) = 0$

that

$$\varphi[p_1(s_{i_1}) p_2(s_{i_2}) \cdots p_m(s_{i_m})] = 0$$

In words: the alternating product of centered variables is centered.

<u>Proof</u>: It suffices to prove the statements for polynomials of the form

$$p_k(s_{i_k}) = s_{i_k}^{p_k} - \varphi(s_{i_k}^{p_k}) \qquad \text{for any power } p_k$$

since general polynomials can be written as linear combinations of those, and the general statement follows then by linearity.
So we have to prove

$$\varphi[(s_{i_1}^{p_1} - \varphi(s_{i_1}^{p_1})\cdot 1)\cdot(s_{i_2}^{p_2} - \varphi(s_{i_2}^{p_2})1) \cdots (s_{i_m}^{p_m} - \varphi(s_{i_m}^{p_m})1)]$$

$$= 0$$

We have LHS =

$$= \sum_{M \subset [m]} (-1)^{|M|} \cdot \prod_{j \in M} \varphi(s_{i_j}^{p_j}) \cdot E[tr_N(\prod_{j \notin M} s_{i_j}^{p_j})]$$

We have now

$$\varphi(s_{i_j}^{p_j}) = \# NC_2(p_j)$$

$$E[tr_N(\prod_{j \notin M} s_{i_j}^{p_j})] = \# NC_2^{[\text{respects indices}]}(\sum_{j \notin M} p_j)$$

Let us put

$I_1 = \{1, \dots, p_1\}$, $\qquad\qquad I = I_1 \cup \dots \cup I_m$

$I_2 = \{p_1 + 1, \dots, p_1 + p_2\}$

$\vdots$

$I_m = \{p_1 + p_2 + \dots + p_{m-1} + 1, \dots, p_1 + p_2 + \dots + p_m\}$

$$\underbrace{\underbrace{\overset{1}{\circ}\ \circ\ \cdots\ \overset{p_1}{\circ}}_{I_1}\ \underbrace{\overset{p_1+1}{\circ}\ \cdots\ \overset{p_1+p_2}{\circ}}_{I_2}\ \circ\ \cdots\ \cdots\ \underbrace{\circ\ \cdots\ \overset{p_1+\dots+p_m}{\circ}}_{I_m}}_{I}$$

then

$$\prod_{j \in M} \varphi(s_{i_j}^{p_j}) \cdot E[tr_N(\prod_{j \notin M} s_{i_j}^{p_j})] =$$

$$= \# \{\pi \in NC_2^{[p]}(I) \mid \forall j \in M: \text{all elements in } I_j \text{ are only paired among themselves}\}$$

Let us denote

$$NC_2^{[p]}(I : j) := \{\pi \in NC_2^{[p]}(I) \mid \text{all elements in } I_j \text{ are only paired among themselves}\}$$

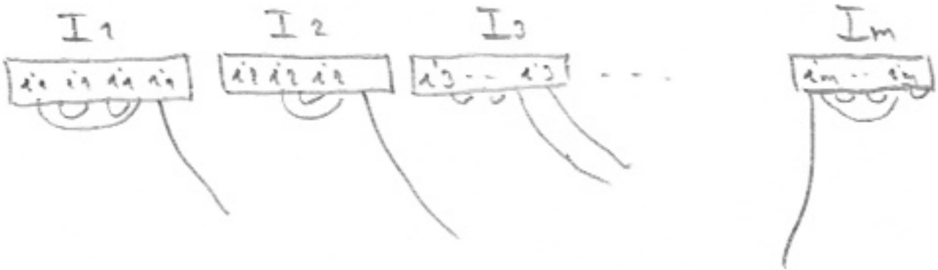
then we have LHS =

$$= \sum_{M \subset [m]} (-1)^{|M|} \cdot \#\left(\bigcap NC_2^{[p]}(I:j)\right)$$

$$= \underbrace{\#\left(NC_2^{[p]}(I) \setminus \bigcup_j NC_2^{[p]}(I:j)\right)}$$ by inclusion-exclusion

this are $\pi \in NC_2^{[p]}(I)$ such that at least one element of each interval I_j is paired with an element from another interval

but there is no such π!



since $i_1 \neq i_2 \neq i_3 \neq \dots \neq i_m$ we cannot connect neighboring intervals, and each interval must be connected to another interval in a non-crossing way. This is not possible, hence

$= 0$ $\square$

13.9. Remarks: 1) Note that we have traded in in 13.8 the explicit description of our moments for implicit relations between moments.

2) For example, the simplest relations from 12.8 are

$$\varphi\left[\left(s_i^p - \varphi(s_i^p)\cdot 1\right)\left(s_j^q - \varphi(s_j^q)\cdot 1\right)\right] = 0 \quad \text{for } i \neq j$$

which can be reformulated to

$$\varphi(s_i^p \cdot s_j^q) - \varphi(s_i^p \cdot 1)\,\varphi(s_j^q) - \varphi(s_i^p)\,\varphi(s_j^q \cdot 1) + \\ + \varphi(s_i^p) \cdot \varphi(s_j^q) \cdot \varphi(1) \quad = 0$$

i.e.,

$$\varphi(s_i^p \cdot s_j^q) = \varphi(s_i^p) \cdot \varphi(s_j^q)$$

Those relations are getting quickly more complicated; e.g.:

$$\varphi[(s_1^{p_1} - \varphi(s_1^{p_1}) \cdot 1)(s_2^{q_1} - \varphi(s_2^{q_1}) \cdot 1)(s_1^{p_2} - \varphi(s_1^{p_2}))(s_2^{q_2} - \varphi(s_2^{q_2}))] = 0$$

leads to

$$\begin{aligned}\varphi[s_1^{p_1} s_2^{q_1} s_1^{p_2} s_2^{q_2}] &= \varphi(s_1^{p_1+p_2})\,\varphi(s_2^{q_1})\,\varphi(s_2^{q_2}) \\ &+ \varphi(s_1^{p_1})\,\varphi(s_1^{p_2})\,\varphi(s_2^{q_1+q_2}) \\ &- \varphi(s_1^{p_1})\,\varphi(s_2^{q_1})\,\varphi(s_1^{p_2})\,\varphi(s_2^{q_2})\end{aligned}$$

3) So one might ask, what is it good for to find those relations between the moments, if we know the moments anyhow in a more explicit form.

Answer: Those relations occur in many more situations; for example independent Wishart matrices satisfy the same relations, though the explicit form of their mixed moments is quite different from the GUE case.

Furthermore, we can control much better what happens with these relations than with the explicit moments, if we deform our setting or construct new random matrices out of other ones.

Not to mention that those relations show also up in very different corners of mathematics (like operator algebras)

To make a long story short: those relations from Theorem 12.8 are really worth to be investigated further, not just in a random matrix context, but also for its own sake

This will be done in the lecture

Free Probability Theory

Freie Wahrscheinlichkeitstheorie

next term!