2. Gaussian Random Matrices: Wick Formula and Combinatorial Proof of Wigner's Semicircle

We want to prove the convergence of our random matrices to the semicircle by showing

$$E[\operatorname{tr}(A_N^{2k})] \xrightarrow{N\to\infty} C_k$$

↑

Catalan numbers

Up to now our matrices were of the form

$$A_N = \frac{1}{\sqrt{N}}(a_{ij})_{i,j=1}^{N}$$

with $a_{ij} = \pm 1$

From an analytic and also combinatorial point of view it is easier to deal with another choice of the a_{ij}, namely we will take them as Gaussian ($\hat{=}$ normal) random variables; and different a_{ij} are still, up to symmetry, independent.

First, we have to understand how to calculate moments of independent Gaussian random variables.

2.1. Def: A standard Gaussian (or normal) random variable is a real-valued Gaussian random variable with mean zero and variance one, i.e. it has distribution

$$P(t_1 \leq X \leq t_2) = \frac{1}{\sqrt{2\pi}} \int_{t_1}^{t_2} e^{-t^2/2} \, dt$$

and hence its moments are given by

$$E[X^n] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^n e^{-t^2/2} \, dt$$

2.2. Proposition: The moments of a standard Gaussian random variable are of the form

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^n e^{-t^2/2} \, dt = \begin{cases} 0 & n \text{ odd} \\ \underbrace{(n-1)(n-3)\ldots 5\cdot 3\cdot 1}_{=:(n-1)!!} & n \text{ ev} \end{cases}$$

Exercise: Show 2.2. by partial integration!

2.3 Remark: From the analytic point of view it is surprising that those integrals are natural numbers. They count actually interesting combinat. objects, namely

$$E[X^{2k}] = \#\{\text{pairings of } 2k \text{ elements}\}$$

2.4. Def.: 1) For a natural number $n \in \mathbb{N}$ we put

$$[n] := \{1, .., n\}$$

2) A *pairing* π of $[n]$ is a decomposition of $[n]$ into disjoint subsets of size 2, i.e.

$\pi = \{V_1, \dots, V_k\}$ where

- $V_i \subset [n] \quad \forall i$
- $\# V_i = 2 \quad \forall i$
- $V_i \cap V_j = \emptyset \quad \forall i \neq j$
- $\bigcup_{i=1}^{k} V_i = [n]$

(note that necessarily $k = n/2$)

3) The set of all pairings of $[n]$ is denoted by

$$\mathcal{P}_2(n) := \{\pi \mid \pi \text{ pairing of } [n]\}$$

2.5. Proposition: 1) We have

$\# \mathcal{P}_2(n) = 0$ if n is odd

and for even $n = 2k$

$$\# \mathcal{P}_2(2k) = (2k-1)(2k-3) \cdots 5 \cdot 3 \cdot 1 = (2k-1)!!$$

2) Hence we have for a standard Gaussian variable X

$$E[X^n] = \# P_2(n)$$

<u>Proof</u>: 1) Count elements in $P_2(n)$ in a recursive way. Choose the pair which contains the element 1; for this we have $n-1$ possibilities. Then we are left with choosing a pairing of the remaining $n-2$ numbers. Hence we have

$$\# P_2(n) = (n-1) \cdot \# P_2(n-2)$$

Iterating this and noting that

$$\# P_2(1) = 0$$

$$\# P_2(2) = 1$$

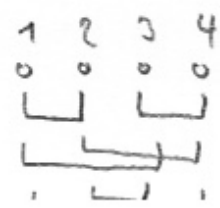
gives the result

2) is a combination of part 1) and 2.2.

<u>2.6. Examples</u>: Usually we draw our partitions by connecting the elements in each pair.

$E[X^2] = 1$ corresponds to

$E[X^4] = 3$ — " —



2.7. Independent Gaussian variables: We will have several Gaussian random variables X, Y and have to calculate mixed moments. The random variables are independent; this means that their joint distribution is the product measure of the single distribution

$$P(t_1 \leq X \leq t_2,\ s_1 \leq Y \leq s_2) =$$

$$P(t_1 \leq X \leq t_2) \cdot P(s_1 \leq Y \leq s_2)$$

so in particular for the moments we have

$$E[X^n \cdot Y^m] = E[X^n] \cdot E[Y^m]$$

This gives also a combinatorial description for their mixed moments:

$$E[X^n \cdot Y^m] = E[X^n] \cdot E[Y^m]$$

$$= \#\{\text{pairings of } \underbrace{X \dots X}_{n}\} \cdot$$

$$\cdot \#\{\text{pairings of } \underbrace{Y \dots Y}_{m}\}$$

$$= \#\{\text{pairings of } \underbrace{X \dots X}_{n}\underbrace{Y \dots Y}_{m}, \text{ which connect } X \text{ with } X \text{ and } Y \text{ with } Y\}$$

Examples: $E[XXYY] = 1$

$$E[XXXYXY] = 3$$

Consider now $x_1, \dots, x_n \in \{X, Y\}$. Then we still have

$$E[x_1 \cdots x_n] = \#\{\text{pairings which connect } X \text{ with } X \text{ and } Y \text{ with } Y\}$$

Can we decide in a more abstract way whether

$x_i = x_j$ or $x_i \neq x_j$

Note that we can read this off from the corresponding second moment, since

$$E[x_i x_j] = \begin{cases} E[x_i^2] = 1 & \text{if } x_i = x_j \\ & \text{(i.e. both } X \text{ or both } Y\text{)} \\ \underbrace{E[x_i]}_{=0} \cdot E[x_j] = 0 & \text{if } x_i \neq x_j \\ & \text{(i.e. one } X\text{, the other } Y\text{)} \end{cases}$$

Hence we have

$$E[x_1,\dots,x_n] = \sum_{\pi\in\mathcal{P}_2(n)} \underbrace{\prod_{(i,j)\in\pi} E[x_i x_j]}_{=:\, E_\pi[x_1,\dots,x_n]}$$

e.g. $E_{\sqcup}[x_1,x_2] = E[x_1\cdot x_2]$

$E_{\sqcup\sqcup}[x_1,x_2,x_3,x_4] = E[x_1\cdot x_2]\,E[x_3x_4]$

$E_{\boxed{\sqcup}}[x_1,x_2,x_3,x_4] = E[x_1x_4]\cdot E[x_2,x_3]$

So we have seen the following.

<u>2.8. Theorem:</u> Let $Y_1,\dots,Y_p$ be independent standard Gaussian random variables and consider $x_1,\dots,x_n\in\{Y_1,\dots,Y_p\}$. Then we have the <u>Wick formula</u>

$$E[x_1\dots x_n] = \sum_{\pi\in\mathcal{P}_2(n)} E_\pi[x_1,\dots,x_n]$$

where we use the notation

$$E_\pi[x_1,\dots,x_n] := \prod_{(i,j)\in\pi} E[x_ix_j]$$

for $\pi\in\mathcal{P}_2(n)$

(Wick 1950, physics
Isserlis 1918, statistics)

Note that the Wick formula is linear in the x_i, hence it remains valid when we replace the x_i by linear combinations of the x_j; in particular, we can go over to complex Gaussian variables.

<u>2.9. Def.</u>: A <u>standard complex Gaussian</u> random variable is of the form

$$Z = \frac{X + iY}{\sqrt{2}},$$

where X and Y are independent standard real Gaussian variables.

<u>2.10. Remark</u>: Let Z be a standard complex Gaussian; i.e. $Z = \frac{X+iY}{\sqrt{2}}$

Then we have

$$\overline{Z} = \frac{X - iY}{\sqrt{2}}$$

and the first and second moments are given by

- $E[Z] = 0 = E[\overline{Z}]$

- $E[Z^2] = E[Z\cdot Z]$

$$= \frac{1}{2}\{\underbrace{E[X\cdot X]}_{=1} - \underbrace{E[Y\cdot Y]}_{=1} + i(\underbrace{E[XY]}_{=0} + \underbrace{E[YX]}_{=0}$$

$$= 0$$

- $E[\bar{Z}^2] = 0$
- $E[|Z|^2] = E[Z\cdot\bar{Z}]$

$$= \frac{1}{2}\{\underbrace{E[X\cdot X]}_{=1} + \underbrace{E[Y\cdot Y]}_{=1} + i(\underbrace{E[YX]}_{=0} - \underbrace{E[XY]}_{=0}$$

$$= 1$$

Hence, for $z_1, z_2 \in \{Z, \bar{Z}\}$, we have

$$E[z_1\cdot z_2] = \begin{cases} 1 & \text{if } z_1\ z_2 \text{ connects } Z \text{ with } \bar{Z} \\ 0 & Z \text{ with } Z \\ & \bar{Z} \text{ with } \bar{Z} \end{cases}$$

and thus the following complex version of the Wick formula.

2.9 Theorem: Let $Z_1, \dots, Z_p$ be independent standard complex Gaussian random variables; consider $z_1, \dots, z_n \in \{Z_1, \bar{Z}_1, \dots, Z_p, \bar{Z}_p\}$

Then we have the Wick formula

$$E[z_1 \dots z_n] = \sum_{\pi \in P_2(n)} E_\pi[z_1, \dots, z_n]$$

$$= \#\{\text{pairings of } [n] \text{ which connect } z_i \text{ with } \bar{z}_i\}$$

Now we are ready to consider random matrices with such complex Gaussians as entries.

2.10. Def.: A Gaussian random matrix is of the form $A_N = \frac{1}{\sqrt{N}} (a_{ij})_{i,j=1}^N$, where

- $A_N = A_N^*$, i.e. $a_{ij} = \bar{a}_{ji}$ $\forall i,j$
- $\{a_{ij} \mid i \geq j\}$ are independent
- each a_{ij} is a standard Gaussian random variable, complex for $i \neq j$ and real for $i = i$

2.11. Remarks: 1) More precisely, we should address the above as a selfadjoint Gaussian random matrix.

2) A common name for these random matrices is also GUE = Gaussian unitary ensemble

"unitary" corresponds here to the fact that the entries are complex, then such matrices are invariant under unitary transformations. There are also real and quaternionic versions,

GOE = Gaussian orthogonal ensemble

GSE = Gaussian symplectic ensemble

3) Note that we can express this definition also in terms of the Wick formula as

$$E[a_{i(1)j(1)} a_{i(2)j(2)} \cdots a_{i(n)j(n)}] =$$

$$= \sum_{\pi \in P_2(n)} E_\pi[a_{i(1)j(1)}, \dots, a_{i(n)j(n)}]$$

$$\forall n; \ 1 \leq i(1), j(1), \dots, i(n), j(n) \leq N$$

and where the second moments are given by

$$E[a_{ij} a_{kl}] = \delta_{il} \delta_{jk}$$

So we have for example for the fourth moment

$$E[a_{i_1j_1} a_{i_2j_2} a_{i_3j_3} a_{i_4j_4}] = \delta_{i_1j_2}\delta_{j_1i_2}\delta_{i_3j_4}\delta_{j_3i_4}$$

$$+ \delta_{i_1j_3}\delta_{j_1i_3}\delta_{i_2j_4}\delta_{j_2i_4}$$

$$+ \delta_{i_1j_4}\delta_{j_1i_4}\delta_{i_2j_3}\delta_{j_2i_3}$$

$$E[a_{12}a_{21}a_{11}a_{11}] = 1$$

$$E[a_{12}a_{12}a_{21}a_{21}] = 2$$

2.12. Calculation of $E[\mathrm{tr}(A_N^m)]$

For our Gaussian random matrices we want to calculate their moments

$$E[\mathrm{tr}(A_N^m)] = \frac{1}{N}\cdot\frac{1}{N^{m/2}}\cdot\sum_{i(1),\dots,i(m)=1}^{N} E[a_{i(1)i(2)}\,a_{i(2)i(3)}\cdots a_{i(m)i(1)}]$$

Let's first consider small examples

1) $$E[\mathrm{tr}(A_N^2)] = \frac{1}{N^2}\sum_{i,j=1}^{N}\underbrace{E[a_{ij}a_{ji}]}_{=1\ \forall i,j}$$

$$= \frac{1}{N^2}\cdot N^2$$

$$= 1$$

hence: $E[\mathrm{tr}(A_N^2)] = 1 = C_1 \qquad \forall N$

(2-1

2) $$E[tr(A_N^4)] = \frac{1}{N^3} \sum_{i,j,k,l=1}^{N} E[a_{ij} a_{jk} a_{kl} a_{li}]$$

$$= \frac{1}{N^3} \sum_{i,j,k,l=1}^{N} E_{\sqcup\sqcup}[\dots] + E_{\sqcup\!\!\sqcup}[\dots] + E_{\sqcup\!\!\!\sqcup}[\dots]$$

$$E_{\sqcup\sqcup}[\underbrace{a_{ij}, a_{jk}}_{i=k}, \underbrace{a_{kl}, a_{li}}_{k=i}] \rightsquigarrow \sum_{\substack{i,j,k,l \\ i=k}} 1 = N^3$$

$$E_{\sqcup\!\!\sqcup}[\underbrace{a_{ij}, \underbrace{a_{jk}, a_{kl}}_{j=l}, a_{li}}_{j=l}] \rightsquigarrow \sum_{\substack{i,j,k,l \\ j=l}} 1 = N^3$$

$$E_{\sqcup\!\!\!\sqcup}[a_{ij}, a_{jk}, a_{kl}, a_{li}] \rightsquigarrow \sum_{\substack{i,j,k,l \\ i=j=k=l}} 1 = N$$

$i=l, j=k$ $\quad$ $j=i, k=l$

$$\Rightarrow E[tr(A_N^4)] = \frac{1}{N^3}(N^3 + N^3 + N)$$

$$= 2 + \frac{1}{N^2}$$

i.e.

$$\lim_{N \to \infty} E[tr(A_N^4)] = 2 = C_2$$

(2-14)

3) Let's now try the general case

$$E[\operatorname{tr}(A_N^m)] =$$

$$= \frac{1}{N^{m/2+1}} \sum_{i_1,\dots,i_m=1}^{N} \underbrace{E[a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_m i_1}]}$$

$$\sum_{\pi \in \mathcal{P}_2(m)} \underbrace{E_\pi[a_{i_1 i_2}, \dots, a_{i_m i_1}]}$$

$$\prod_{(k,l)\in\pi} \underbrace{E[a_{i_k i_{k+1}} a_{i_l i_{l+1}}]}$$

$$[i_k = i_{l+1}]$$
$$[i_{k+1} = i_l]$$

(we use here notation $\delta_{ij} \hat{=} [i=j]$)

$$= \frac{1}{N^{m/2+1}} \sum_{\pi \in \mathcal{P}_2(m)} \sum_{i_1,\dots,i_m=1}^{N} \underbrace{\prod_k [i_k = i_{\underbrace{\pi(k)+1}_{\gamma\pi(k)}}]}_{}$$

where
$\gamma = (1\,2 \dots m) \in S_m$
shift by 1

we identify pairing π
with a permutation $\pi \in S_m$

$(k,l) \in \pi \iff \pi(k) = l$
$\pi(l) = k$

this is $\neq 0$ if and only if
$i : [m] \to [N]$
is constant on the cycles
of $\gamma\pi \in S_m$

$$= \frac{1}{N^{m/2+1}} \sum_{\pi \in P_2(m)} N^{\#(\gamma\pi)}$$

where $\#(\gamma\pi)$ = number of cycles of $\gamma\pi$

Hence we have proved the following

<u>2.13. Theorem:</u> Let A_N be a Gaussian (GUE) $N \times N$ random matrix. Then we have for all $m \in \mathbb{N}$

$$E[tr(A_N^m)] = \sum_{\pi \in P_2(m)} N^{\#(\gamma\pi) - \frac{m}{2} - 1}$$

<u>2.14. Examples:</u> 1) all odd moments are zero, since $P_2(2k+1) = \emptyset$

2) $m = 2$, $\gamma = (1\ 2)$

$\pi = (1\ 2) \Rightarrow \gamma\pi = id = (1)(2)$

$\Rightarrow \#(\gamma\pi) = 2$

$\Rightarrow \#(\gamma\pi) - \frac{m}{2} + 1 = 2 - 2 = 0$

$\Rightarrow E[tr(A_N^2)] = N^0 = 1$

3) $m = 4$, $\gamma = (1\ 2\ 3\ 4)$

Then there are three contributions

$\pi \in P_2(4)$

π	$\gamma\pi$	$\#(\gamma\pi)-3$	contribu
$(12)(34)$	$(13)(2)(4)$	0	$N^0 = 1$
$(13)(24)$	(1432)	-2	$N^{-2} = 1/$
$(14)(23)$	$(1)(24)(3)$	0	$N^0 = 1$

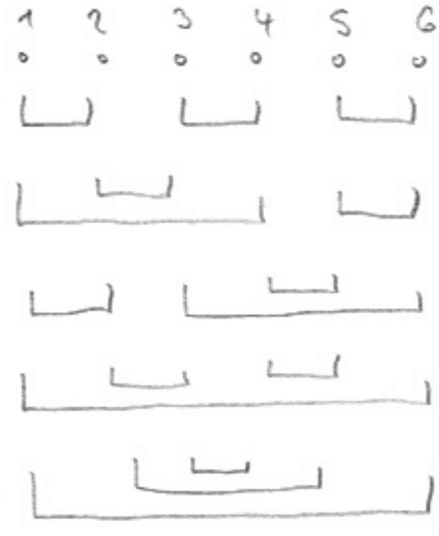
$$\Rightarrow E[tr(A_N^4)] = 2 + \frac{1}{N^2}$$

4) In the same way

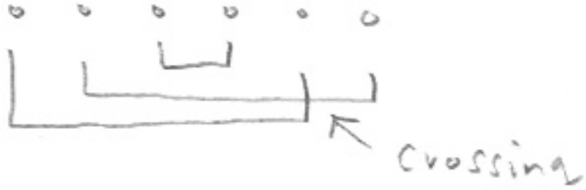
$$E[tr(A_N^6)] = 5 + 10\,\frac{1}{N^2}$$

$$E[tr(A_N^8)] = 14 + 70\,\frac{1}{N^2} + 21\,\frac{1}{N^4}$$

5) For $m=6$ the following 5 pairings give the contribution N^0:

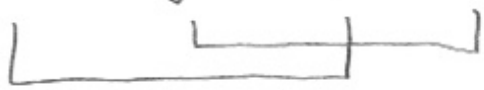


Those are non-crossing pairings, all other $\pi \in \mathcal{P}_2(6)$ have a crossing, like



2.15. Definition: A pairing $\pi \in \mathcal{P}_2(m)$ is <u>non-crossing</u> if there are no pairs (i,k) and (j,l) in π with $i<j<k<l$; i.e. we don't have a crossing in π

$$i < j < k < l$$



We put

$$NC_2(m) := \{\pi \in \mathcal{P}_2(m) \mid \pi \text{ is non-crossing}\}$$

2.16. Examples: i) $m=2$, $NC_2(2) = \mathcal{P}_2(2) =$

$= \{$  $\}$

ii) $m=4$

$NC_2(4) = \{$ ,  $\}$

$\mathcal{P}_2(4) \setminus NC_2(4) = \{$  $\}$

iii) $m=6$

5 Elements in $NC_2(6)$ are given in 2.14 (5); there are $15-5=10$ more elements in $\mathcal{P}_2(6)$ with crossings, e.g.

 has maximal $(=3)$ number of crossings

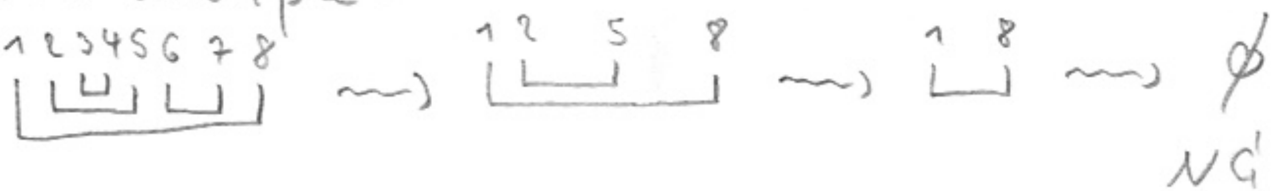
<u>2.17. Remarks</u>: Note that NC-pairings have a recursive structure, which is usually crucial for dealing with them

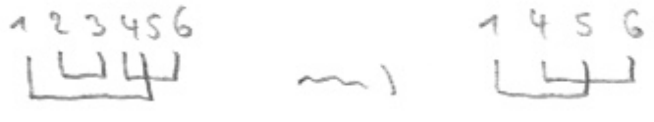
1) The first pair of $\pi \in NC_2(2k)$ must necessarily be of the form $(1, 2l)$ and the remaining pairs can only pair within $\{2, .., 2l-1\}$ or within $\{2l+1, .., 2k$

2) Iterating this shows that we must find in $\pi \in NC_2(2k)$ at least one pair of the form $(i, i+1)$; removing this pair gives a NC-pairing of $2k-2$ points.

This characterises NC-pairings as those which can be reduced to the empty set by an iterated removal of i $i+1$

For example:





no further removal possible

2.18. Proposition: Consider m even and let $\pi \in \mathcal{P}_2(m)$, which we identify with a permutation $\pi \in S_m$. As before, $\gamma := (1\ 2 \dots m) \in S_m$. Then we have

(i) $\#(\gamma\pi) - \frac{m}{2} - 1 \leq 0 \qquad \forall \pi \in \mathcal{P}_2(m)$

(ii) $\#(\gamma\pi) - \frac{m}{2} - 1 = 0 \iff \pi \in NC_2(m)$

Proof: First we note that a pair $\overset{i\ \ i+1}{\sqcup}$ in π corresponds to a fixed point of $\gamma\pi$, namely

more precisely

if $\pi = \cdots \overset{i-1}{\circ}\ \underset{\sqcup}{\overset{i}{\circ}\ \overset{i+1}{\circ}}\ \overset{i+2}{\circ} \cdots$, then

$$\gamma\pi: \quad i+1 \xrightarrow{\pi} i \xrightarrow{\gamma} i+1$$
$$\qquad\quad i \xrightarrow{\pi} i+1 \xrightarrow{\gamma} i+2$$

note that $\overset{i\ \ i+1}{\sqcup}$ means here actually $\overset{i\ \ \gamma(i)}{\sqcup}$ i.e. 1 and m count also as neighbors

hence $\gamma\pi$ has cycles $(i+1)$ and $(\dots i, i+2 \dots)$

This implication goes also in the other direction: if $\gamma\pi(i+1) = i+1$

$$\Downarrow$$

$$\pi(i+1) = \gamma^{-1}(i+1) = i$$

Since π is a pairing we must have then also $\pi(i) = i+1$ and

$\Rightarrow$ hence we have the pair $\overset{i\ \ i+1}{\sqcup}$ in π

If we have such a situation $\underset{\sqcup}{i\ \ i+1}$ in π, we can remove the points i and $i+1$, yielding another partition $\tilde{\pi}$. By doing so we remove in $\gamma\pi$ the cycle $(i+1)$ and remove in the cycle $(\ldots i, i+2, \ldots)$ the point i yielding $\gamma\tilde{\pi}$. We reduce thus m by 2 and $\#(\gamma\pi)$ by 1.

If π is NC we can iterate this until we arrive at $\tilde{\pi}$ with $m = 2$; then we have one pair

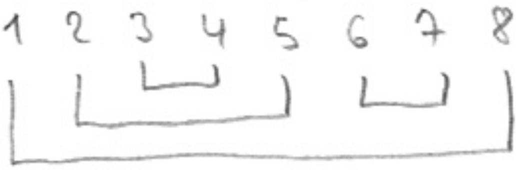
$\tilde{\pi} = \underset{\sqcup}{1\ \ 2}$, $\gamma = (1,2) \Rightarrow \gamma\tilde{\pi} = (1)(2)$

with $\#(\gamma\tilde{\pi}) = 2$

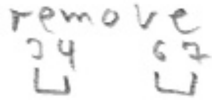
If $m = 2k$ we did $k-1$ reductions where we reduced the # of cycles by 1 and at the end we remain with 2 cycles, hence

$$\#(\gamma\pi) = (k-1)\cdot 1 + 2 = k+1 = \frac{m}{2}+1$$

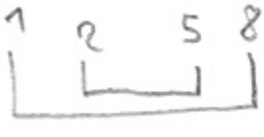
here is an example for this:

$\pi =$ 

$\gamma\pi = (1)(2\,\cancel{6}\,8)(\cancel{3}\,5)(\cancel{4})($

$\downarrow$ remove 

$\downarrow$



$(1)(\cancel{2}\,8)(\cancel{5})$

$$\downarrow \text{ remove } \overset{2\ 5}{\sqcup} \qquad\qquad\qquad \downarrow$$

$$\overset{1 \qquad 8}{\sqcup\!\!\!\sqcup} \qquad\qquad\qquad \cancel{(1)}\,\cancel{(8)}$$

$$\downarrow \text{ remove } \overset{1\ 8}{\sqcup} \qquad\qquad\qquad \downarrow$$

$$\emptyset \qquad\qquad\qquad\qquad 0$$

For a general $\pi \in \mathcal{P}_2(m)$ we remove $\overset{i\ i+1}{\sqcup}$ as long as possible; if π is crossing we arrive at a pairing $\tilde{\pi}$ when this is not possible any more. It suffices now to show that such a $\tilde{\pi} \in \mathcal{P}_2(m)$ satisfies

$$\#(\gamma\tilde{\pi}) - \frac{m}{2} - 1 < 0$$

But since $\tilde{\pi}$ has no pair $\overset{i\ i+1}{\sqcup}$, $\gamma\tilde{\pi}$ has no fixed point, hence each cycle has at least 2 elements, and thus

$$\#(\gamma\tilde{\pi}) \leq \frac{m}{2} < \frac{m}{2} + 1$$

$\square$

With this we get from 2.13.

<u>2.14. Theorem (Wigner's semicircle law for GUE, averaged version)</u>: Let A_N be a Gaussian (GUE) $N \times N$ random matrix.

Then we have for all $m \in \mathbb{N}$:

$$\lim_{N\to\infty} E[tr(A_N^m)] = \frac{1}{2\pi}\int_{-2}^{+2} x^m \sqrt{4-x^2}\, dx$$

<u>Proof:</u> This is true for m odd because then both sides are zero.

Consider $m = 2k$ even. Then 2.13 and 2.18 show that

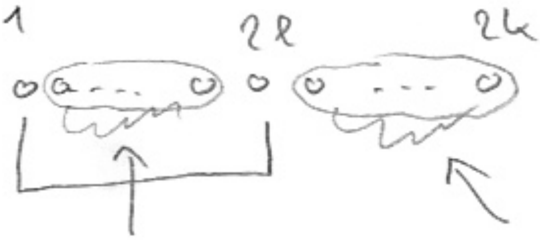
$$\lim_{N\to\infty} E[tr(A_N^m)] = \sum_{\pi\in P_2(m)} \underbrace{\lim_{N\to\infty} N^{\#(\gamma\pi)-\frac{m}{2}-1}}_{=\begin{cases}1 & \pi\in NC_2(m)\\ 0 & \pi\notin NC_2(m)\end{cases}}$$

$$= \# NC_2(m)$$

Since the moments of the semicircle are given by the Catalan numbers it remains to see that $\# NC_2(2k) = C_k$

Let us put $\# NC_2(2k) =: d_k$

We count $\# NC_2(2k)$ now according to the recursive structure of NC-pairings as in 2.17 (1)



NC pairing of $2(l-1)$ elements

NC pairing of $2(k-l)$ elements

We can identify a $\pi \in NC_2(2k)$ with

$(1, 2l) \cup \pi_0 \cup \pi_1$

when $l \in \{1, \dots, k\}$, $\pi_0 \in NC_2(2(l-1))$
$\pi_1 \in NC_2(2(k-l))$

hence we have

$$d_k = \sum_{l=1}^{k} d_{l-1} \cdot d_{k-l} \qquad \text{where } d_0 := 1$$

This is the recursion for the Catalan numbers; since also $d_0 = 1$, we have that $d_k = C_k$ $\square$

<u>2.15 Remarks:</u> 1) One can refine

$$\#(\gamma\pi) - \frac{m}{2} - 1 \leq 0$$

to

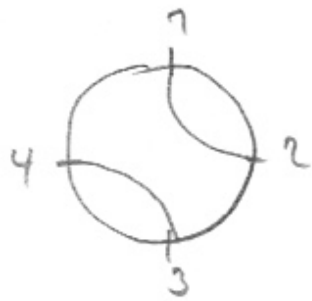
$$\#(\gamma\pi) - \frac{m}{2} - 1 = -2g(\pi)$$

for $g(\pi) \in \mathbb{N}_0$. This g has then the meaning that it is the minimal genus of a surface, on which π can be drawn without crossings. In this terminology a NC pairing is also called "planar", corresponding to $g = 0$.

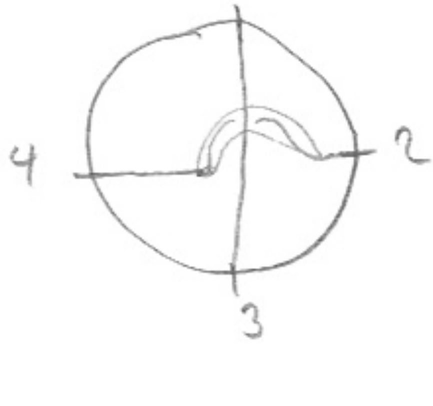
2.13 is usually addressed as genus expansion

$$E[tr(A_N^m)] = \sum_{\pi \in P_2(m)} N^{-2g(\pi)}$$

2) For example, $(12),(34) \in NC_2(4)$ has $g = 0$



but the crossing $(13),(24)$ has $g=1$, it has a crossing in the plane, but this can be avoided on a torus



3) If we denote

$$\varepsilon_g(k) := \#\{\pi \in P_2(2k) \mid \pi \text{ has genus } g\},$$

then 2.13 can be written as

$$E[\operatorname{tr}(A_N^{2k})] = \sum_{g \geq 0} \varepsilon_g(k) N^{-2g}$$

We know that

$$\varepsilon_0(k) = C_k = \frac{1}{k+1}\binom{2k}{k}$$

What about the other $\varepsilon_g(k)$ for $g>0$?

There does not exist an explicit formula for them, but Harer and Zagier have shown in 1986 that

$$\varepsilon_g(k) = \frac{(2k)!}{(k+1)!(k-2g)!} \times$$

$$\times \text{ coefficient of } x^{2g} \text{ in } \left(\frac{x/2}{\tanh x/2}\right)^{n+1}$$