5. Analytic Proof of Wigners Semi-circle Law for Gaussian Random Matrices

5.1. Remark: We consider now real Gaussian random matrices (GOE) of the form $A_N = (x_{ij})_{i,j=1}^N$,
where $x_{ij} = x_{ji}$ and $\{x_{ij} \mid i \leq j\}$ are i.i.d variables with Gaussian distribution of mean zero and variance $\frac{1}{N}$
$E[x_{ij}] = 0$, $E[x_{ij}^2] = \frac{1}{N}$.
More formally, on the space of symmetric $N \times N$-matrices

$$\Omega_N = \left\{ A_N = (x_{ij})_{i,j=1}^N \mid x_{ij} \in \mathbb{R},\ x_{ij} = x_{ji}\ \forall i,j \right\}$$

we consider the probability measure

$$dP_N(A_N) = C_N \cdot \exp\left(-\frac{N}{4} \operatorname{Tr}(A_N^2)\right) \prod_{i \leq j} dx_{ij}$$

[Note that with this P_N, which is invariant under orthogonal rotations, we have actually different variances on and off the diagonal:
$E[x_{ij}^2] = \frac{1}{N}$ $(i \neq j)$ and $E[x_{ii}^2] = \frac{2}{N}$]

We consider now, for each $N \in \mathbb{N}$, the averaged eigenvalue distribution

$$\mu_N := E[\mu_{A_N}] = \int_{\Omega_N} \mu_A \, dP_N(A_N)$$

We want to prove that

$$\mu_N \xrightarrow{w} \mu_w$$

According to 4.16 we can prove this by showing that for all $z \in \mathbb{C}^+$:

$$\lim_{N \to \infty} S_{\mu_N}(z) = S_{\mu_w}(z).$$

Note that

$$S_{\mu_N}(z) = \int_{\mathbb{R}} \frac{1}{t-z} \, d\mu_N(t)$$

$$= E[\underbrace{\int_{\mathbb{R}} \frac{1}{t-z} \, d\mu_A(t)}_{S_{\mu_{A_N}}(t) \underset{\text{by Ass 4, Ex 1}}{=} tr[(A_N - z \cdot 1)^{-1}]}]$$

$$= E[tr[(A_N - z \cdot 1)^{-1}]]$$

So what we have to see, is for $z \in \mathbb{C}^+$:

$$\lim_{N \to \infty} E[tr[(A_N - z \cdot 1)^{-1}]] = S_{\mu_w}(z)$$

Let us denote $R_{A_N}(z) = \frac{1}{A_N - z \cdot 1}$,

so that $S_{\mu_N}(z) = E[\mathrm{tr}(R_{A_N}(z))]$

We want to see that $S_{\mu_N}(z)$ satisfies approximately the quadratic equation of S_μ

We have: $(A - z \cdot 1) \cdot R_A(z) = 1$

$$\Rightarrow A \cdot R_A(z) - z \cdot R_A(z) = 1$$

$$\Rightarrow R_A(z) = -\frac{1}{z} + \frac{1}{z} \cdot A\, R_A(z)$$

$$\Rightarrow E[\mathrm{tr}(R_A(z))] = -\frac{1}{z} + \frac{1}{z} E[\underbrace{\mathrm{tr}(A \cdot R_A(z))}_{\frac{1}{N}\sum_{k,l} x_{kl} (R_A(z))_{lk}}]$$

Gaussian variable

function of independent Gaussian variables

<u>5.2. Proposition (Stein's identity)</u>: Let $X_1, \dots, X_k$ be independent random variables with Gaussian distribution, with mean zero and variances $E[X_i] = \sigma_i^2$. Let $h : \mathbb{R}^k \to \mathbb{C}$ be continuously differentiable such that h and all partial derivatives are of polynomial growth.

Then we have for $i = 1, \dots, k$:

$$E[X_i\, h(X_1, \dots, X_k)] = \sigma_i^2\, E\left[\frac{\partial h}{\partial x_i}(X_1, \dots, X_k)\right].$$

More explicitely,

$$\int_{\mathbb{R}^k} x_i\, h(x_1, \dots, x_k)\, e^{-\frac{x_1^2}{2\sigma_1^2}} \cdots e^{-\frac{x_k^2}{2\sigma_k^2}}\, dx_1 \dots dx_k$$

$$= \sigma_i^2 \int_{\mathbb{R}^k} \frac{\partial h}{\partial x_i}(x_1, \dots, x_k)\, e^{-x_1^2/2\sigma_1^2} \cdots e^{-x_k^2/2\sigma_k^2}\, dx_1 \dots dx_k$$

<u>Proof</u>: For $k = 1$, we have

$$\int_{\mathbb{R}} x\, h(x)\, e^{-x^2/2\sigma} dx = \int_{\mathbb{R}} h(x)\, \underbrace{x\, e^{-x^2/2\sigma}}_{[-\sigma^2 \cdot e^{-x^2/2\sigma^2}]'}\, dx$$

$$\overset{\text{partial integration}}{=} \int_{\mathbb{R}} h'(x)\, \sigma^2 e^{-x^2/2\sigma}\, dx$$

(Assumptions on h are such that the boundary terms vanish!)

For general k, we just do partial integration for the i-th coordinate $\square$

<u>5.3 Remark:</u> We want to apply this now to our Gaussian random matrices, with Gaussian random variables x_{ij} $(1 \le i \le j \le N)$ of variance

$$\sigma_{ij}^2 = \begin{cases} 1/N & i \neq j \\ 2/N & i = j \end{cases}$$

and for the function

$$h(x_{ij} \mid i \leq j) = h(A) = [R_A(z)]_{l,k}$$

$$\text{where } R_A(z) = \frac{1}{A - z\cdot 1}$$

<u>5.4. Lemma</u>: For $A = (x_{ij})_{i,j=1}^{N}$, with $x_{ij} = x_{ji}$ for all i,j, we have for all i,j,k,l

$$\frac{\partial}{\partial x_{ij}} [R_A(z)]_{lk} =$$

$$= \begin{cases} -[R_A(z)]_{li} [R_A(z)]_{ik} & i = j \\ -[R_A(z)]_{li} [R_A(z)]_{jk} - [R_A(z)]_{lj} [R_A(z)]_{ik} & i \neq j \end{cases}$$

<u>Proof</u>: Note first that

$$\frac{\partial}{\partial x_{ij}} A = \begin{cases} E_{ii} & i = j \\ E_{ij} + E_{ji} & i \neq j \end{cases}$$

where E_{ij} is matrix unit with 1 at position (i,j) and 0 elsewhere

We have $R_A(z) \cdot (A - z\cdot 1) = 1$,

which yields by differentiating

$$\frac{\partial R_A(z)}{\partial x_{ij}} \cdot (A - z \cdot 1) + R_A(z) \cdot \frac{\partial A}{\partial x_{ij}} = 0$$

$$\Rightarrow \frac{\partial R_A(z)}{\partial x_{ij}} = - R_A(z) \cdot \frac{\partial A}{\partial x_{ij}} R_A(z)$$

thus: for $i=j$

$$\left[\frac{\partial R_A(z)}{\partial x_{ii}} \right]_{lk} = - \left[R_A(z) \cdot E_{ii} \cdot R_A(z) \right]_{lk}$$

$$= - [R_A(z)]_{li} \, [R_A(z)]_{ik}$$

for $i \neq j$

$$\left[\frac{\partial R_A(z)}{\partial x_{ij}} \right]_{lk} = - \left[R_A(z) \, E_{ij} \, R_A(z) \right]_{lk} - \left[R_A(z) \cdot E_{ji} \cdot R_A(z) \right]_{lk}$$

$$= [R_A(z)]_{li} \, [R_A(z)]_{jk} - [R_A(z)]_{lj} \cdot [R_A(z)]_{ik}$$

$\square$

<u>5.5. Theorem</u>: Let A_N be GOE random matrices as in 5.1. Then its averaged eigenvalue distribution $\mu_N := E[\mu_{A_N}]$ converges weakly to the semicircle distribution: $\mu_N \xrightarrow{w} \mu_W$

<u>Proof</u>: By 4.16, it suffices to show

$$\lim_{N\to\infty} S_{\mu_N}(z) = S_{\mu_W}(z) \qquad \forall z \in \mathbb{C}^+$$

We also know (see 5.1.) that

$$S_{\mu_N}(z) = E[\operatorname{tr}[(A_N - z\cdot 1)^{-1}]]$$

As before, we write

$$R_A(z) := (A_N - z\cdot 1)^{-1}$$

and thus have, as in 5.1.,

$$\underbrace{E[\operatorname{tr}(R_A(z))]}_{= S_{\mu_N}(z)} = -\frac{1}{z} + \frac{1}{z} E[\operatorname{tr}(A R_A(z))]$$

Now we calculate, with $A = (x_{ij})_{i,j=1}^{N}$,

$$E[\operatorname{tr}(A R_A(z))] = \frac{1}{N} \sum_{k,l=1}^{N} E[x_{kl} [R_A(z)]_{lk}]$$

$$\overset{(5.2)}{=} \frac{1}{N} \sum_{k,l=1}^{N} \sigma_{kl}^2 \underbrace{E\left[\frac{\partial}{\partial x_{kl}} [R_A(z)]_{lk}\right]}$$

$$\overset{(5.4)}{=} -\frac{1}{N}\left([R_A(z)]_{lk}[R_A(z)]_{lk} + [R_A(z)]_{ll}[R_A(z)]_{kk}\right)$$

(note that $\sigma_{kl}^2 = \begin{cases} 1/N & k \neq l \\ 2/N & k = l \end{cases}$)

Note that $(A_N - z \cdot 1)$ is symmetric,
hence the same is true for its inverse

$$R_A(z) = (A_N - z \cdot 1)^{-1}$$

and thus: $[R_A(z)]_{\ell k} = [R_A(z)]_{k\ell}$

So we get

$$E[tr(A R_A(z))] =$$

$$= \underbrace{-\frac{1}{N} E[tr(R_A(z)^2)]}_{\substack{\text{this should} \\ \text{go to zero} \\ (*)}} - \underbrace{E[tr(R_A(z)) \cdot tr(R_A(z))]}_{\substack{\text{this should be close to} \\ E[tr(R_A(z))] \cdot E[tr(R_A(z))] \\ = S_{\mu_N}(z)^2 \\ (**)}}$$

(*) A_N as symmetric matrix can be diagonalized by an orthogonal matrix U,

$$A_N = U \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} U^*$$

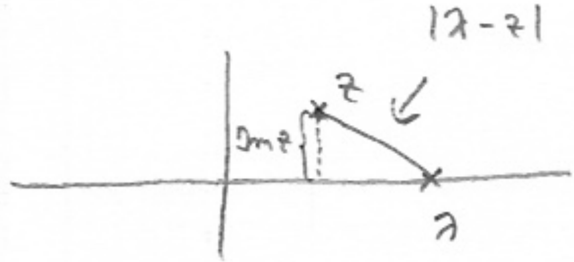
and thus

$$R_A(z)^2 = U \begin{pmatrix} \frac{1}{(\lambda_1 - z)^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{(\lambda_N - z)^2} \end{pmatrix} U^*$$

$$\Rightarrow \ |\operatorname{tr}[R_A(z)^2]| \leq \frac{1}{N} \sum_{i=1}^{N} \left| \frac{1}{(\lambda_i - z)^2} \right|$$

Note that for all $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}^+$:

$$\left| \frac{1}{\lambda - z} \right| \leq \frac{1}{\operatorname{Im} z}$$



and thus

$$\frac{1}{N} |E[\operatorname{tr}(R_A(z)^2)]| \leq \frac{1}{N} E[\underbrace{|\operatorname{tr}[R_A(z)^2]|}_{\leq \frac{1}{\operatorname{Im} z^2}}]$$

$$\longrightarrow 0 \quad \text{for } N \to \infty$$

(**) By definition of the variance we have

$$\begin{aligned} \operatorname{Var}(X) &:= E[(X - E[X])^2] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

and thus

$$E[X^2] = E[X]^2 + \operatorname{Var}(X)$$

Hence in (**) we can replace

$$E[\operatorname{tr}(R_A(z)) \cdot \operatorname{tr}(R_A(z))]$$ by

$$\underbrace{E[tr(R_A(z))]^2}_{= S_{\mu_N}(z)^2} + Var(\underbrace{tr(R_A(z))}_{S_{\mu_A}(z)})$$

In the next section we will show that we have concentration, i.e. the variance $Var(S_{\mu_A}(z))$ goes to zero for $N \to \infty$.

With this we have then

$$S_{\mu_N}(z) = -\frac{1}{z} - \frac{1}{z} S_{\mu_N}(z)^2 + \varepsilon_N \qquad (***)$$

where $\varepsilon_N \to 0$ for $N \to \infty$

Note that, as above, for any Stieltjes transform S_ν, we have

$$|S_\nu(z)| = \left| \int \frac{1}{t-z} d\nu(t) \right|$$

$$\leq \int \underbrace{\left|\frac{1}{t-z}\right|}_{\leq \frac{1}{Im\, z}} d\nu(t) \leq \frac{1}{Im\, z}$$

and thus $(S_{\mu_N}(z))_N$ is a bounded sequence of complex numbers. Hence, by compactness, there exists a convergent subsequence $(S_{\mu_{N(m)}}(z))_m$ which converges to some $S(z)$

This $S(z)$ must then satisfy the limit $N \to \infty$ of equation (***), thus

$$S(z) = -\frac{1}{z} - \frac{1}{z} S(z)^2$$

Since all $S_{\mu_N}(z)$ are in $\mathbb{C}^+$, the limit $S(z)$ must be in $\overline{\mathbb{C}^+}$, which leaves for $S(z)$ only the possibility that

$$S(z) = \frac{-z + \sqrt{z^2 - 4}}{2} \qquad \left(\begin{array}{l}\text{as the other solution} \\ \text{is in } \mathbb{C}^-\end{array}\right)$$

$$= S_{\mu_W}(z)$$

In the same way it follows that any subsequence of $(S_{\mu_N}(z))_N$ has a convergent subsequence which converges to $S_{\mu_W}(z)$; this forces all cluster points of $(S_{\mu_N}(z))_N$ to be $S_{\mu_W}(z)$, and thus the whole sequence converges to $S_{\mu_W}(z)$.

This implies then: $\mu_N \xrightarrow{w} \mu_W$ $\square$

To complete the proof we still have to see the concentration in (**) !