7. Analytic Description of the Eigenvalue Distribution of Gaussian RM

In Ass 2., Ex 4 we showed that the joint distribution of the entries $a_{ij} = x_{ij} + y_{ij} \cdot i$ of a GUE $A = (a_{ij})_{i,j=1}^{N}$ has density

$$c \cdot \exp\left[-\frac{N}{2} \operatorname{Tr}(A^2)\right] dA$$

This shows clearly the invariance of this distribution under unitary transformation.
Let U be a unitary $N \times N$ matrix and put

$$B := U A U^* = (b_{ij})_{i,j=1}^{N}$$

Since $\operatorname{Tr}(B^2) = \operatorname{Tr}(A^2)$
and the volume element is invariant under unitary transformations, $dB = dA$, we have

$$\underbrace{c \cdot e^{-\frac{N}{2}\operatorname{Tr}(A^2)} dA}_{\substack{\text{distribution of} \\ \text{entries of } A}} = \underbrace{c \cdot e^{-\frac{N}{2}\operatorname{Tr}(B^2)} dB}_{\substack{\text{distribution of} \\ \text{entries of } B}}$$

thus: joint distribution of entries of GUE is invariant under unitary transformations

GUE = Gaussian unitary ensemble

What we are interested in are not the entries but the eigenvalues of our matrices. Thus we should transform this density from entries to eigenvalues.

Instead of GUE we consider first the real version, GOE.

7.1. Def.: A Gaussian orthogonal (GOE) random matrix $A = (x_{ij})_{i,j=1}^{N}$ is given by real-valued entries with $x_{ij} = x_{ji}$ $\forall i,j$ and joint distribution

$$c_N \exp\left[-\frac{N}{4} \operatorname{Tr}(A^2)\right] \underset{\substack{\uparrow \\ \prod_{i \geq j} dx_{ij}}}{dA}$$

6.2. Remarks: 1) This is clearly invariant under orthogonal transformations of the entries.

2) This is equivalent to independent real Gaussian random variables; note however that the variance on the diagonal has to be chosen different from the off-diagonal. Check this for $N = 2$:

$$\exp\left[-\frac{N}{4}\operatorname{Tr}\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22}\end{pmatrix}^2\right] =$$

$$= \exp\left[-\frac{N}{4}\left(x_{11}^2 + x_{12}^2 + x_{12}^2 + x_{22}^2\right)\right]$$

$$= \exp\left[-\frac{N}{4}x_{11}^2\right]\cdot\exp\left[-\frac{N}{2}x_{12}^2\right]\cdot\exp\left[-\frac{N}{4}x_{22}^2\right]$$

Gaussian of variance $\frac{1}{N}$

Gaussian of variance $\frac{2}{N}$

Hence we have for $GOE(N)$ $A = (x_{ij})_{i,j=1}^N$

$$E[x_{ij}^2] = \begin{cases} \frac{1}{N} & i \neq j \\ \frac{2}{N} & i = j \end{cases}$$

3) From this one can easily determine the normalization constant c_N (as a function of N). This will be done in an exercise. $\square$

Since we are usually interested in functions of the eigenvalues, we will now transform this density to eigenvalues.

7.3. Example: Let us first consider the case $N=2$ for GOE: We have

$$A = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \quad \text{with density}$$

$$p(A) = c_2 \cdot e^{-\frac{N}{4}\operatorname{Tr}(A^2)}$$

We parametrise A by its eigenvalues λ_1, λ_2 and an angle Θ by diagonalising:

$A = O^t D O$ where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad O = \underbrace{\begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}}$$

orthogonal 2×2-matrix (with $\det = 1$)

i.e.:
$$x_{11} = \lambda_1 \cos^2\Theta + \lambda_2 \sin^2\Theta$$
$$x_{12} = (\lambda_1 - \lambda_2)\cos\Theta \cdot \sin\Theta$$
$$x_{22} = \lambda_1 \sin^2\Theta + \lambda_2 \cdot \cos^2\Theta$$

note: O and D are not uniquely determined by A

in particular: if $\lambda_1 = \lambda_2$, then any orthogonal O works; however this case has probability zero, thus can be ignored (→ next remark)

If $\lambda_1 \neq \lambda_2$, then we can choose $\lambda_1 < \lambda_2$;
O contains then the normed eigenvectors for λ_1 and λ_2; those are unique up to sign, which can be fixed by requiring that $\cos\theta \geq 0$; hence θ is not running from $-\pi$ to π, but is restricted to $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Let us now transform

$$p(x_{11}, x_{12}, x_{22})\, dx_{11}\, dx_{12}\, dx_{22} \to q(\lambda_1, \lambda_2, \theta)\, d\lambda_1\, d\lambda_2\, d\theta$$

By the change of variables formula (Transformationssatz from Analysis III) we have

$$q = p \cdot \text{Jacobian}$$

$$\uparrow$$

$$J = |\det DF| \quad \text{where}$$

$$F: (x_{11}, x_{12}, x_{22}) \to (\lambda_1, \lambda_2, \theta)$$

$$\Rightarrow q(\lambda_1, \lambda_2, \theta) = c_2 \cdot e^{-\frac{N}{4}\operatorname{Tr}(A^2)} \cdot J$$

$$= c_2 \cdot e^{-\frac{N}{4}(\lambda_1^2 + \lambda_2^2)} \cdot J$$

What is J?

$$DF = \begin{pmatrix} \cos^2\theta & \sin^2\theta & -2(\lambda_1-\lambda_2)\sin\theta\cos\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & (\lambda_1-\lambda_2)[-\sin^2\theta+\cos^2\theta] \\ \sin^2\theta & \cos^2\theta & 2(\lambda_1-\lambda_2)\sin\theta\cos\theta \end{pmatrix}$$

$$\Rightarrow \det DF = (\lambda_1-\lambda_2)\begin{vmatrix} \cos^2\theta & 1 & -2\sin\theta\cos\theta \\ \cos\theta\sin\theta & 0 & \cos^2\theta-\sin^2\theta \\ \sin^2\theta & 1 & 2\sin\theta\cos\theta \end{vmatrix}$$

$$= (\lambda_1-\lambda_2)\begin{vmatrix} \cos^2\theta & 1 & -2\sin\theta\cos\theta \\ \cos\theta\sin\theta & 0 & \cos^2\theta-\sin^2\theta \\ 1 & 2 & 0 \end{vmatrix}$$

$$= (\lambda_1-\lambda_2)\,[\cos^2\theta-\sin^2\theta\underbrace{-4\sin^2\theta\cos^2\theta - 2\cos^4\theta + 2\cos^2\theta\sin^2\theta}_{-\cos^2\theta-\sin^2\theta\ =\ -1}]$$

$$\Rightarrow J = |\lambda_1-\lambda_2|$$

thus: $$q(\lambda_1,\lambda_2,\theta) = c_2\cdot e^{-\frac{N}{4}(\lambda_1^2+\lambda_2^2)}\,|\lambda_1-\lambda_2|$$

note: q is independent of θ, i.e. in θ we have uniform distribution

Consider a function f of eigenvalues,
$f = f(\lambda_1, \lambda_2)$; then

$$E[f] = \iiint q(\lambda_1, \lambda_2, \theta)\, f(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2 d$$

$$= \iint_{\lambda_1 < \lambda_2} \int_{-\frac{\pi}{2}}^{\pi/2} f(\lambda_1, \lambda_2)\, c_2\, e^{-\frac{N}{4}(\lambda_1^2 + \lambda_2^2)}\, |\lambda_1 - \lambda_2|$$

$$d\lambda_1 d\lambda_2 d\theta$$

$$= \iint_{\lambda_1 < \lambda_2} f(\lambda_1, \lambda_2) \cdot \underbrace{\pi c_2}_{= \tilde{c}_2} e^{-\frac{N}{4}(\lambda_1^2 + \lambda_2^2)} |\lambda_1 - \lambda_2| \, d\lambda_1 d\lambda$$

Thus: the density for the joint distribution of the eigenvalues on $\{(\lambda_1, \lambda_2) \mid \lambda_1 < \lambda_2\}$ is given by

$$\tilde{c}_2\, e^{-\frac{N}{4}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)$$

<u>7.4. Remark</u>: Let us check that the probability of $\lambda_1 = \lambda_2$ is zero!

λ_1, λ_2 are solutions of characteristic equation

$$\det(\lambda 1 - A) = 0$$

$$\begin{vmatrix} \lambda - x_{11} & -x_{12} \\ -x_{12} & \lambda - x_{22} \end{vmatrix} = 0$$

i.e. $(\lambda - x_{11})(\lambda - x_{22}) - x_{12}^2 = 0$

i.e. $\lambda^2 - (x_{11} + x_{22})\lambda + x_{11}x_{22} - x_{12}^2 = 0$

Two solutions coincide

$\iff$ discriminant $= 0$

$$\text{discriminant} = (x_{11}+x_{22})^2 - 4(x_{11}x_{22} - x_{12}^2) = 0$$

however: $\{(x_{11}, x_{12}, x_{22}) \mid \ldots = 0\}$

is two-dimensional surface in $\mathbb{R}^3$, i.e. its Lebesgue measure is zero.

Now we consider general GOE(N).

<u>7.5 Theorem:</u> The joint distribution of the eigenvalues of a GOE(N) is given by a density

$$\tilde{c}_N \; e^{-\frac{N}{4}(\lambda_1^2 + \ldots + \lambda_N^2)} \prod_{k<l} (\lambda_l - \lambda_k),$$

restricted on $\lambda_1 < \lambda_2 < \ldots < \lambda_N$

<u>Proof:</u> In terms of entries we have density

$$p(x_{kl} \mid k \geq l) = c_N \cdot e^{-\frac{N}{4}\mathrm{Tr}(A^2)}$$

where $A = (x_{kl})_{k,l=1}^N$ with x_{kl} real and $x_{kl} = x_{lk}$ $\forall l,k$

Again we want to diagonalize

$$A = O^t D O$$

D: diagonal, O: orthogonal

As before, degenerated eigenvalues have probability zero, hence this case can be neglected. So we can assume

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \quad \text{with } \lambda_1 < \lambda_2 < \dots < \lambda_N$$

We parametrize O now via $O = e^{-H}$ by a skew-symmetric matrix H, with $H^t = -H$

i.e. $H = (h_{ij})_{i,j=1}^N$ with $h_{ij} = -h_{ji}$, $h_{ij} \in \mathbb{R}$

in particular $h_{ii} = 0 \quad \forall i$

note: $O^t = e^{-H^t} = e^H$, thus $O^t O = 1 = O O^t$

and O is orthogonal

Note that this parametrization has the right number of parameters:

$$A \mathrel{\hat{=}} \{x_{ij} \mid i \geq j\} \mathrel{\hat{=}} \frac{N(N+1)}{2} \text{ coordinates}$$

$$O^t D O \mathrel{\hat{=}} \{\lambda_1, \dots, \lambda_N\} \cup \{h_{ij} \mid i > j\}$$

$$\mathrel{\hat{=}} N + \frac{N^2 - N}{2} = \frac{N(N+1)}{2} \text{ coordinates}$$

This parametrization is locally bijective.
[$O = e^{-H}$ is parametrization of Lie group $SO(N)$ by Lie algebra $so(N)$ of skew-symmetric matrices.]

So we need Jacobian of the map

$$S: A \mapsto e^{H} D e^{-H}$$

We have

$$dA = de^{H} D e^{-H} + e^{H} dD e^{-H} + e^{H} D de^{-H}$$

$$= e^{H} \{ e^{-H} de^{H} D - D de^{-H} e^{H} + dD \} e^{-H}$$

this transports the calculation of the derivative at arbitrary point e^{H} to the identity element $1 = e^{0}$ in Lie group; since Jacobian is preserved under this transformation, it suffices to calculate the Jacobian at $H = 0$, i.e. for $e^{H} = 1$, $de^{H} = dH$, when it is

$$dH \cdot D - D \cdot dH + dD$$

i.e. $dx_{ij} = dh_{ij} \lambda_j - \lambda_i dh_{ij} + d\lambda_i \delta_{ij}$ or

$$\frac{\partial x_{ij}}{\partial \lambda_k} = \delta_{ij} \delta_{ik} \; , \quad \frac{\partial x_{ij}}{\partial h_{kl}} = \delta_{ik} \delta_{jl} (\lambda_l - \lambda_k)$$

hence: $J = |DF| =$

$$
\begin{array}{c|cc|}
 & x_{ii} & x_{ij}\ (i<j) \\
\begin{matrix}\lambda_1 \\ \vdots \\ \lambda_N\end{matrix} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 \\
\begin{matrix}h_{kl} \\ (k<l)\end{matrix} & 0 & \begin{matrix}\ddots & & \\ & \lambda_l-\lambda_k & \\ & & \ddots\end{matrix}
\end{array}
= \prod_{k<l} (\lambda_l - \lambda_k)
$$

Thus we have

$$
\begin{aligned}
q(\lambda_1,\dots,\lambda_N; h_{kl}) &= p(x_{ij} \mid i<j) \cdot J \\
&= c_N \underbrace{e^{-\frac{N}{4}\operatorname{Tr}(A^2)}}_{e^{-\frac{N}{4}(\lambda_1^2+\dots+\lambda_N^2)}} \cdot \prod_{k<l} (\lambda_l - \lambda_k)
\end{aligned}
$$

This is independent of the "angles" h_{kl}; integrating over those just changes the constant c_N to $\tilde{c}_N$ and gives the result $\square$

In a similar way the complex case can be treated (Exercise!); there one gets the following.

7.6 Theorem: The joint distribution of the eigenvalues of a GUE(N) is given by a density

$$\hat{c}_N \, e^{-\frac{N}{2}(\lambda_1^2 + \dots + \lambda_N^2)} \prod_{k<l} (\lambda_l - \lambda_k)^2,$$

restricted on $\lambda_1 < \lambda_2 < \dots < \lambda_N$.

7.7. Definition: The function

$$\Delta(\lambda_1, \dots, \lambda_N) := \prod_{\substack{k,l=1 \\ k<l}}^{N} (\lambda_l - \lambda_k)$$

is called Vandermonde determinant.

7.8. Proposition: For $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ we have

$$\Delta(\lambda_1, \dots \lambda_N) = \det(\lambda_j^{i-1})_{i,j=1}^{N}$$

$$= \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_N^2 \\ \vdots & & & \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix}$$

Proof: $\det(\lambda_j^{i-1})$ is polynomial in $\lambda_1, \dots, \lambda_N$

If $\lambda_l = \lambda_k$ for $k < l$, then $\det(\lambda_j^{i-1}) = 0$

Thus: $\det(\lambda_j^{i-1})$ contains a factor

$\lambda_\ell - \lambda_k$ for each $k < \ell$,

hence Δ divides $\det(\lambda_j^{i-1})$

Since $\det(\lambda_j^{i-1})$ is a sum of products with one factor from each row, we have

$$\deg \det(\lambda_j^{i-1}) = 0 + 1 + 2 + \dots + (N-1)$$
$$= \frac{N(N-1)}{2}$$
$$= \deg \Delta$$

$\Rightarrow \Delta(\lambda_1, \dots, \lambda_N) = c \det(\lambda_j^{i-1})$ for some $c \in \mathbb{R}$

By comparing the coefficient of $1 \cdot \lambda_2 \cdot \lambda_3^2 \cdots \lambda_N^{N-1}$ on both sides one checks that $c = 1$. $\square$

Note: in $\det(\lambda_j^{i-1})$ we can add to the k-th row arbitrary linear combinations of smaller rows without changing the value of the determinant; i.e. we can replace λ^k by an arbitrary monic polynomial of degree k

$$p_k(\lambda) = \lambda^k + \alpha_{k-1}\lambda^{k-1} + \dots + \alpha_1 \lambda + \alpha_0$$

Hence we have the following proposition

7.9. Proposition: Let $p_0, \dots, p_{N-1}$ be monic polynomials p_k of degree k. Then we have

$$\det\left(p_{i-1}(\lambda_j)\right)_{i,j=1}^{N} = \Delta(\lambda_1, \dots \lambda_N)$$

$$= \prod_{k<l} (\lambda_l - \lambda_k)$$

In the following we will make a special choice for the p_k; we choose them as Hermite polynomials, which are orthogonal with respect to the Gauss distribution $\frac{1}{c} e^{-\frac{1}{2}\lambda^2}$

7.10 Def.: The Hermite polynomials $H_n(x)$ are defined by

i) H_n is monic polynomial of degree n

ii) We have for all $n, m \geq 0$:

$$\int_{\mathbb{R}} H_n(x) H_m(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \delta_{nm}\, n!$$

7.11 Remarks: 1) One can get the $H_n(x)$ from the monomials $1, x, x^2, x^3, \dots$ via Gram-Schmidt process as follows.

We define an inner product on $C(\mathbb{R})$ by

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \bar{g}(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

We put • $H_0(x) := 1$; monic of degree 0

and $\langle H_0, H_0 \rangle = \int \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = 1 = 0!$

• $H_1(x) := x$, monic of degree 1

and $\langle H_1, H_0 \rangle = \frac{1}{\sqrt{2\pi}} \int x e^{-x^2/2} dx = 0$

$\langle H_1, H_1 \rangle = \frac{1}{\sqrt{2\pi}} \int x^2 e^{-x^2/2} dx = 1 = 1!$

• for H_2, note that

$\langle x^2, H_1 \rangle = \frac{1}{\sqrt{2\pi}} \int x^3 e^{-x^2/2} dx = 0$

but

$\langle x^2, H_0 \rangle = \frac{1}{\sqrt{2\pi}} \int x^2 e^{-x^2/2} dx = 1$

hence we set

$H_2(x) := x^2 - H_0 = x^2 - 1$

Then we have

$\langle H_2, H_0 \rangle = 0 = \langle H_2, H_1 \rangle$

and

$$\langle H_2, H_2 \rangle = \frac{1}{\sqrt{2\pi}} \int \underbrace{(x^2-1)^2}_{x^4 - 2x^2 + 1} e^{-x^2/2} dx$$

$$= 3 - 2 + 1$$

$$= 2 = 2!$$

· continue in this way

Note that the H_n are uniquely determined by the requirements that H_n is monic and that $\langle H_n, H_m \rangle = 0 \quad \forall n \neq m$.

That we have then $\langle H_n, H_n \rangle = n!$ is a statement which has to be proved.

2) The Hermite polynomials satisfy many explicit relations; important is the three-term recurrence

$$x\, H_n(x) = H_{n+1}(x) + n\, H_{n-1}(x) \qquad \forall n \geq 1$$

4) The first few H_n are

$$H_0(x) = 1$$

$$H_1(x) = x$$

$$H_2(x) = x^2 - 1$$

$$H_3(x) = x^3 - 3x$$

$$H_4(x) = x^4 - 6x^2 + 3$$

etc ..

5) By 7.9, we can now use the H_n for writing our Vandermonde,

$$\Delta(\lambda_1, \dots \lambda_N) = \det\left(H_{i-1}(\lambda_j)\right)_{i,j=1}^{N}.$$

We want to use this for our GUE(N) density:

$$q(\lambda_1,\dots,\lambda_N) = \hat{c}_N\, e^{-\frac{N}{2}(\lambda_1^2+\dots+\lambda_N^2)}\, \Delta(\lambda_1,\dots,\lambda_N)^2$$

$$= \hat{c}_N\, e^{-\frac{1}{2}(\mu_1^2+\dots+\mu_N^2)}\, \underbrace{\Delta\left(\tfrac{\mu_1}{\sqrt{N}},\dots,\tfrac{\mu_N}{\sqrt{N}}\right)}_{\Delta(\mu_1,\dots,\mu_N)\cdot\left(\frac{1}{\sqrt{N}}\right)^{N(N-1)}}{}^2$$

where $\mu_1=\sqrt{N}\lambda_1,\dots,\mu_N=\sqrt{N}\lambda_N$ are the eigenvalues of the "unnormalized" GUE matrix $\sqrt{N}A_N$. It will be easier to deal with those; we will now also go over from ordered eigenvalues $\{\lambda_1<\lambda_2<\dots<\lambda_n\}$ to unordered $(\mu_1,\dots,\mu_N)\in\mathbb{R}^N$; since in the latter case each ordered tuple shows up $N!$ times, this gives an additional factor $N!$ in our density, we collect all these N-dependent factors in our constant $\tilde{c}_N$ (we don't loose any information by this, since we did not know the explicit form of $\hat{c}_N$).

So we have now the density

$$p(\mu_1,\dots,\mu_N) = \tilde{c}_N\, e^{-\frac{1}{2}(\mu_1^2+\dots+\mu_N^2)}\, \Delta(\mu_1,\dots,\mu_N)^2$$

$$= \tilde{c}_N\, e^{-\frac{1}{2}(\mu_1^2+\dots+\mu_N^2)} \left(\det\left(H_{i-1}(\mu_j)\right)_{i,j=1}^N\right)^2$$

$$= \tilde{c}_N \left[\det \left(e^{-\frac{1}{4}\mu_j^2} \cdot H_{i-1}(\mu_j) \right)_{i,j=1}^{N} \right]^2$$ (6-

7.12 Def.: The Hermite functions Ψ_n are defined as

$$\Psi_n(x) = (2\pi)^{-1/4} (n!)^{-1/2} e^{-\frac{1}{4}x^2} H_n(x)$$

7.13 Remarks: 1) We have

$$\int_{\mathbb{R}} \Psi_n(x) \Psi_m(x) \, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n!m!}} \cdot \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} H_n(x) H_m(x) \, dx$$

$$= \delta_{nm}$$

i.e. the Ψ_n are orthonormal with respect to Lebesgue measure; actually they form a orthonormal bases of $L^2(\mathbb{R})$.

2) Now we can continue the calculation of $p(\mu_1, \dots \mu_N)$ as follows

$$p(\mu_1, \dots, \mu_N) = \underset{\substack{\uparrow \\ \text{again a new constant}}}{c_N} \left[\det \left(\Psi_{i-1}(\mu_j) \right)_{i,j=1}^{N} \right]^2$$

Denote $V_{ij} = \Psi_{i-1}(\mu_j)$. Then we have

$$(\det V)^2 = (\det V^t) \cdot \det V = \det(V^t V)$$

and we have

$$(V^t \cdot V)_{ij} = \sum_{k=1}^{N} V_{ki} V_{kj}$$

$$= \sum_{k=1}^{N} \Psi_{k-1}(\mu_i)\, \Psi_{k-1}(\mu_j)$$

<u>7.14. Definition:</u> The N^{th} <u>Hermite kernel</u> is

$$K_N(x,y) = \sum_{k=0}^{N-1} \Psi_k(x)\, \Psi_k(y)$$

We have thus proved the following.

<u>7.15. Theorem:</u> The unordered joint eigenvalue distribution of an unnormalized $GUE(N)$ is given by a density

$$p(\mu_1, \dots, \mu_N) = c_N \det\left(K_N(\mu_i, \mu_j)\right)_{i,j=1}^{N},$$

where K_N is the Hermite kernel.

<u>7.16. Proposition:</u> K_N is a <u>reproducing kernel</u>, i.e. we have

$$\int K_N(x,u)\, K_N(u,y)\, du = K_N(x,y)$$

<u>Proof:</u> $\int k_N(x,u) \cdot k_N(u,y)\, du =$

$$= \int \sum_{k=0}^{N-1} \Psi_k(x)\, \Psi_k(u) \cdot \sum_{l=0}^{N-1} \Psi_l(u) \cdot \Psi_l(y)\, du$$

$$= \sum_{k,l=0}^{N-1} \Psi_k(x)\, \Psi_l(y) \underbrace{\int \Psi_k(u)\, \Psi_l(u)\, du}_{=\delta_{kl}}$$

$$= \sum_{k=0}^{N-1} \Psi_k(x)\, \Psi_k(y)$$

$$= k_N(x,y) \qquad \square$$

<u>7.17. Lemma:</u> Let $k : \mathbb{R}^2 \to \mathbb{R}$ be a reproducing kernel, i.e.

$$\int k(x,u)\, k(u,y)\, du = k(x,y)$$

Put $d := \int k(x,x)\, dx$.

[We assume that all those integrals make sense, as e.g. for our k_N.]

Then we have for all $n \geq 2$

$$\int \det\big(k(\mu_i,\mu_j)\big)_{i,j=1}^{n}\, d\mu_n =$$

$$(d-n+1) \det\big(k(\mu_i,\mu_j)\big)_{i,j=1}^{n-1}$$

Proof: $n = 2$:

$$\underbrace{\int \det \begin{pmatrix} k(\mu_1,\mu_1) & k(\mu_1,\mu_2) \\ k(\mu_2,\mu_1) & k(\mu_2,\mu_2) \end{pmatrix}}_{k(\mu_1,\mu_1)k(\mu_2,\mu_2) - k(\mu_1,\mu_2)k(\mu_2,\mu_1)} d\mu_2$$

$$= k(\mu_1,\mu_1) \underbrace{\int k(\mu_2,\mu_2) d\mu_2}_{d} - \underbrace{\int k(\mu_1,\mu_2) k(\mu_2,\mu_1) d\mu_2}_{k(\mu_1,\mu_1)}$$

$$= (d-1)\, k(\mu_1,\mu_1)$$

$$= (d-1) \det\left(k(\mu_1,\mu_1)\right)$$

$n = 3$:

$$\begin{vmatrix} k(1,1) & k(1,2) & k(1,3) \\ k(2,1) & k(2,2) & k(2,3) \\ k(3,1) & k(3,2) & k(3,3) \end{vmatrix} =$$

$$= \begin{vmatrix} k(2,1) & k(2,2) \\ k(3,1) & k(3,2) \end{vmatrix} k(1,3) \quad \leftarrow 1$$

$$- \begin{vmatrix} k(1,1) & k(1,2) \\ k(3,1) & k(3,2) \end{vmatrix} k(2,3) \quad \leftarrow 2$$

$$+ \begin{vmatrix} k(1,1) & k(1,2) \\ k(2,1) & k(2,2) \end{vmatrix} k(3,3) \quad \leftarrow 3$$

$$\int ③ \, d\mu_3 = \begin{vmatrix} k(1,1) & k(1,2) \\ k(2,1) & k(2,2) \end{vmatrix} \cdot d$$

$$\int ② \, d\mu_3 = - \int \begin{vmatrix} k(1,1) & k(1,2) \\ k(2,3)k(3,1) & k(2,3)k(3,2) \end{vmatrix} d\mu$$

$$= - \begin{vmatrix} k(1,1) & k(1,2) \\ k(2,1) & k(2,2) \end{vmatrix}$$

$$\int ① \, d\mu_3 = \int \begin{vmatrix} k(2,1) & k(2,2) \\ k(1,3)k(3,1) & k(1,3)k(3,2) \end{vmatrix} d\mu$$

$$= \begin{vmatrix} k(2,1) & k(2,2) \\ k(1,1) & k(1,2) \end{vmatrix}$$

$$= - \begin{vmatrix} k(1,1) & k(1,2) \\ k(2,1) & k(2,2) \end{vmatrix}$$

$$\Rightarrow \int \det\left(k(\mu_i,\mu_j)\right)_{i,j=1}^{3} d\mu_3 =$$

$$= (d-2) \det\left(k(\mu_i,\mu_j)\right)_{i,j=1}^{2}$$

general n works in the same way

Iteration of 7.17 gives then

7.18 Corollary: Under the assumptions of 7.17 we have

$$\int \dots \int \det \left(k(\mu_i, \mu_j) \right)_{i,j=1}^{n} d\mu_1 \dots d\mu_n$$

$$= (d-n+1)(d-n+2) \dots (d-1)\, d$$

7.19 Remark: We want to apply this now to our Hermite kernel, i.e. $k = k_N$.

Then we have

$$d = \int k_N(x,x)\, dx$$

$$= \int \sum_{k=0}^{N-1} \Psi_k(x) \Psi_k(x)\, dx$$

$$= \sum_{k=0}^{N-1} \underbrace{\int \Psi_k(x) \Psi_k(x)\, dx}_{=1} = N$$

and thus

$$\int \dots \int \det \left(k_N(\mu_i, \mu_j) \right)_{i,j=1}^{N} d\mu_1 \dots d\mu_N$$

$$= N!$$

as now $d = N = n$

This allows now to determine our constant c_N in the density $p(\mu_1, \dots, \mu_N)$ from 7.15. Since p is a probability density on $\mathbb{R}^N$ we have

$$1 = \int_{\mathbb{R}^N} p(\mu_1, \dots, \mu_N)\, d\mu_1 \dots d\mu_N$$

$$= c_N \underbrace{\int_{\mathbb{R}^N} \det\left(k_N(\mu_i, \mu_j)\right)_{i,j=1}^{N} d\mu_1 \dots d\mu_N}_{= N!}$$

hence

$$c_N = \frac{1}{N!}$$

So we have proved the following more precise version of 7.15

<u>7.20 Theorem:</u> The unordered joint eigenvalue distribution of an unnormalised $GUE(N)$ is given by a density

$$p(\mu_1, \dots, \mu_N) = \frac{1}{N!} \det\left(k_N(\mu_i, \mu_j)\right)_{i,j=1}^{N}$$

where k_N is the Hermite kernel

$$k_N(x, y) = \sum_{k=0}^{N-1} \Psi_k(x)\, \Psi_k(y)$$

7.22. Theorem: The averaged eigenvalue density of an unnormalised $GUE(N)$ is given by

$$= \frac{1}{N} k_N(\mu,\mu)$$

$$\rho_N(\mu) = \frac{1}{N} \sum_{k=0}^{N-1} \psi_k(\mu)^2$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{k!} H_k(\mu)^2 e^{-x^2/2}$$

Proof: Note that

$$p(\mu_1, \dots \mu_N)\, d\mu_1 \dots d\mu_N$$

is probability to have N eigenvalues at

$$[\mu_1, \mu_1 + d\mu_1] \times \dots \times [\mu_N, \mu_N + d\mu_N]$$

If we are integrating out $N-1$ variables we are left with the probability for one eigenvalue (without caring about the others)

Hence

$$\rho_N(\mu) = \int p(\mu_1, \dots, \mu_{N-1}, \mu)\, d\mu_1 \dots d\mu_{N-1}$$

$$= \frac{1}{N!} \int \det\left(k_N(\mu_i, \mu_j)\right)_{i,j=1}^{N} d\mu_1 \dots d\mu_{N-1}$$

$$(\mu_N := \mu)$$

$$= \frac{1}{N!} (N-1)! \det\left(k_N(\mu,\mu)\right) = \frac{1}{N} k_N(\mu,\mu)$$

$\square$