8. Determinental processes and non-crossings paths: Karlin-McGregor and Gessel-Viennot

8.1. Remark: Our probability distributions for the eigenvalues of GUE have a determinental structure, i.e. are of form

$$p(\mu_1, \dots, \mu_N) = \frac{1}{N!} \det \left(K_N(\mu_i, \mu_j) \right)_{i,j=1}^{N} ; \quad (*)$$

they describe N eigenvalues which repel each other (via the factor $(\mu_i - \mu_j)^2$).
If we consider corresponding processes, then the paths of the eigenvalues should not cross.

There is a quite general relation between determinants as in $(*)$ and non-crossing paths; this appeared independently in different contexts:

i) Karlin - McGregor 1958:
Markov chains and Brownian motions

ii) Lindström 1973: "matroids"

iii) Gessel - Viennot 1985 : combinatorics

8.2. Stochastic version à la Karlin - McGregor

Consider a random walk on integers $\mathbb{Z}$

Y_k = position at time k

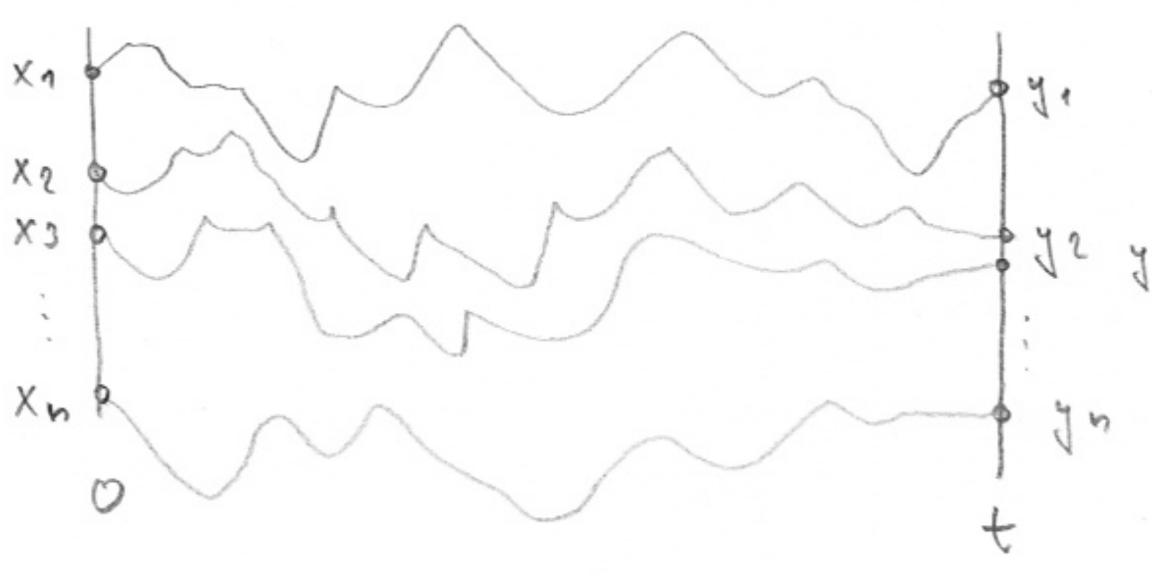
$\mathbb{Z} \hat{=}$ possible positions

transition probability (to the two neighbours) might depend on position, i.e.

$$i-1 \xleftarrow{\;q_i\;} i \xrightarrow{\;p_i\;} i+1 \qquad q_i + p_i = 1$$

We consider now n copies of such a random walk, which start at different positions.
at time $k=0$

We are interested in the probability that the paths don't cross.



Let x_i be such that all distances are even, i.e. if two paths cross they have to meet at some time.

8.3 Theorem (Karlin - McGregor):

Consider n copies of Y_k, i.e. $(Y_k^{(1)}, \dots, Y_k^{(n)})$ with $Y_0^{(i)} = x_i$ where

$$x_1 > x_2 > \dots > x_n$$

Consider now $y_1 > y_2 > \dots > y_n$ and $t \in \mathbb{N}$

We put

$$P_t(x_i, y_j) = P\{Y_t = y_j \mid Y_0 = x_i\}$$

= probability of one random walk to get from x_i to y_j in t steps

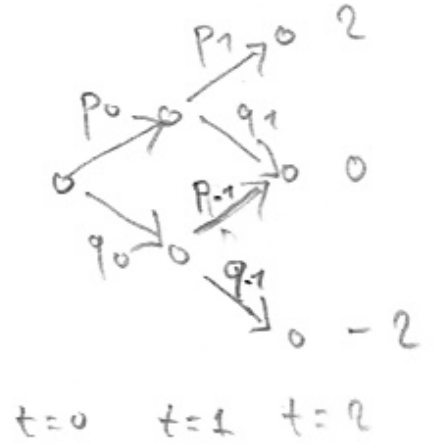
Then we have

$$P(Y_t^{(i)} = y_i \ \forall i, \ Y_s^{(1)} > Y_s^{(2)} > \dots > Y_s^{(n)} \ \forall\, 0 \le \underset{\substack{\cap \\ \mathbb{N}}}{s} \le t$$

$$= \det\left(P_t(x_i, y_j)\right)_{i,j=1}^{n}$$

8.4. Example: For one random walk Y_t (7-

we have



$p_0 p_1 = \frac{1}{4}$

$p_0 q_1 + q_0 p_{-1} = \frac{1}{2}$

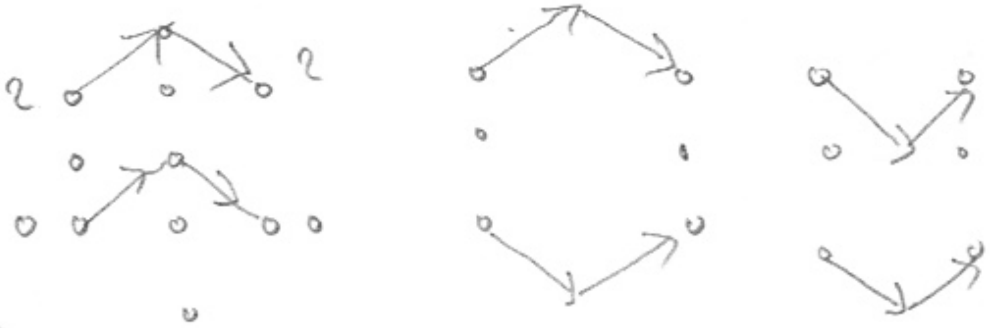
$q_0 q_{-1} = \frac{1}{4}$

for $p_0 = q_0 = \frac{1}{2}$, $p_{\pm 1} = q_{\pm 1} = \frac{1}{2}$ symmetric random walk

Consider now two such symmetric random walks and set $x_1 = 2 = y_1$, $x_2 = 0 = y_2$; then

$$P\left(Y_2^{(1)} = 2 = Y_0^{(1)},\ Y_2^{(2)} = 0 = Y_0^{(2)},\ Y_1^{(1)} > Y_2^{(1)}\right.$$

$= P\Big\{$

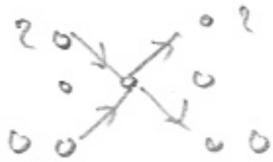


$\Big\}$

1/16 1/16 1/16

$= 3/16$

Note that



is not allowed

8.3. says that we get this result also as

$$\det \begin{array}{c} \begin{array}{cc} 0 & 2 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 2 \end{array} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

<u>Proof</u>: Let Ω_{ij} be the set of all possible paths in t steps from x_i to y_j; denote by $P(\pi)$ the probability for such a path $\pi \in \Omega_{ij}$. Then we have

$$P_t(x_i, y_j) = \sum_{\pi \in \Omega_{ij}} P(\pi)$$

and we have to consider the determinant

$$\det\left(P_t(x_i, y_j)\right)_{i,j=1}^{n} =$$

$$= \det\Big(\sum_{\pi \in \Omega_{ij}} P(\pi)\Big)_{i,j=1}^{n}$$

Let us consider case $n = 2$:

$$\det\begin{pmatrix} \sum\limits_{\pi \in \Omega_{11}} P(\pi) & \sum\limits_{\pi \in \Omega_{12}} P(\pi) \\ \sum\limits_{\pi \in \Omega_{21}} P(\pi) & \sum\limits_{\pi \in \Omega_{22}} P(\pi) \end{pmatrix} =$$

$$= \underbrace{\sum_{\pi \in \Omega_{11}} P(\pi) \cdot \sum_{\sigma \in \Omega_{22}} P(\sigma)}_{(i)} - \underbrace{\sum_{\pi \in \Omega_{12}} P(\pi) \cdot \sum_{\sigma \in \Omega_{21}} P(\sigma)}_{(ii)}$$

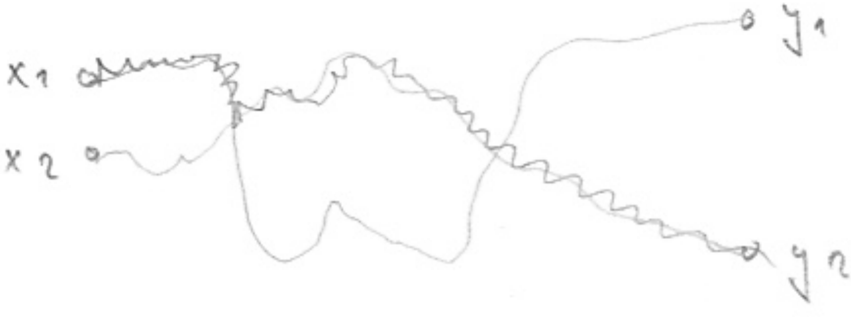
(i) counts all pairs of paths from

$x_1 \to y_1$ and $x_2 \to y_2$;

hence non-crossing ones

but also crossing ones

however, such a crossing pair of paths is via the "reflection principle" in bijection with a pair of paths from $x_1 \to y_2$ and $x_2 \to y_1$



(where the bijection preserves the probabilities)

Those paths, $x_1 \to y_2$ and $x_2 \to y_1$, are counted by the term (ii); hence (ii) cancels all crossing terms in (i) and we remain only with the non-crossing paths.

General n works in the same way $\square$

8.5 Combinatorial version à la Gessel-Viennot

Let G be a weighted graph without cycles

e.g.

where we have weights on each edge

$$i \xrightarrow{e} j \qquad m_{ij} = m_e$$

this gives weights for paths

P  $\quad m(P) = \prod_{e \in P} m_e$

and then also a weight for connecting two vertices a, b

$$m(a,b) = \sum_{P : a \to b} m(P)$$

sum of the weights of all paths connecting a to b.

Consider now two n-tuples of vertices

$A = (a_1, \dots a_n)$ and $B = (b_1, \dots, b_n)$

A path system $P : A \to B$ is given by a $\sigma \in S_n$ and paths $P_1 : a_1 \to b_{\sigma(1)}$,

$P_2 : a_2 \to b_{\sigma(2)}, \dots, P_n : a_n \to b_{\sigma(n)}$

and we also put $\sigma(P) = \sigma$ and

$\operatorname{sign}(P) := \operatorname{sign}(\sigma)$

A vertex-disjoint path system is a path system $(P_1, \ldots, P_n)$, where the paths $P_1, \ldots, P_n$ don't have a common vertex

8.6. Lemma of Gessel-Viennot: Let G be a finite acyclic weighted directed graph, and $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ two n-sets of vertices. Then we have

$$\det\left(m(a_i, b_j)\right)_{i,j=1}^{n} =$$

$$= \sum_{\substack{P: A \to B \\ \text{vertex-disjoint}}} \operatorname{sign}(\sigma(P)) \cdot \prod_{i=1}^{n} m(P_i)$$

Proof: similar as for 8.3, the crossing paths cancel in the determinant $\square$

8.7. Example: Let C_n be the Catalan numbers $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, ... and consider

$$M_n = \begin{pmatrix} C_0 & C_1 & C_2 & \cdots & C_n \\ C_1 & C_2 & C_3 & \cdots & \vdots \\ C_2 & C_3 & C_4 & \cdots & \vdots \\ \vdots & & \vdots & & \\ C_n & & \cdots & & C_{2n} \end{pmatrix}$$

Then we have

$$\det M_0 = |C_0| = 1$$

$$\det M_1 = \begin{vmatrix} C_0 & C_1 \\ C_1 & C_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

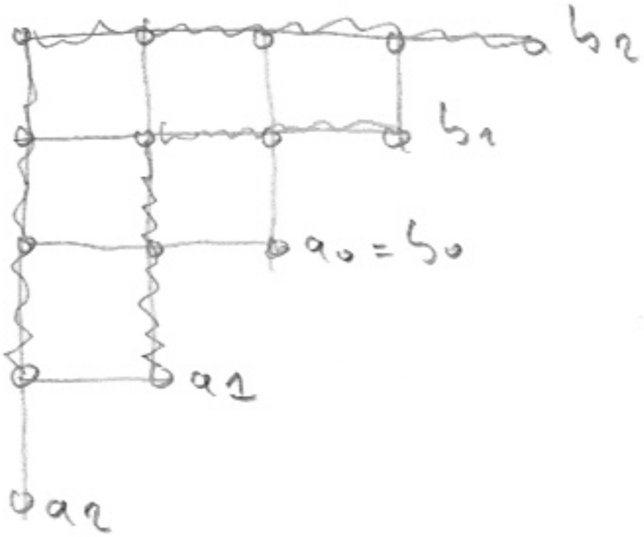
$$\det M_2 = \begin{vmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{vmatrix}$$

$$= 28 + 10 + 10 - 8 - 14 - 25$$

$$= 1$$

This is actually true for all n: $\det M_n = 1$.
This follows from Gessel-Viennot.

Let us show it for M_2; for this consider the graph



directions $\uparrow \rightarrow$
all weights $= 1$

Catalan numbers count Dyck paths in this graph, i.e.
$m(a_0, b_0) = C_0$
$m(a_0, b_1) = C_1$ etc.

hence

$$M_2 = \begin{pmatrix} m(a_0, b_0) & m(a_0, b_1) & m(a_0, b_2) \\ m(a_1, b_0) & m(a_1, b_1) & m(a_1, b_2) \\ m(a_2, b_0) & m(a_2, b_1) & m(a_2, b_2) \end{pmatrix}$$

But then Gessel-Viennot says that

$$\det M_2 = \sum_{\substack{\text{vertex-disjoint} \\ P : (a_0, a_1, a_2) \to (b_0, b_1, b_2)}} \cdots$$

there is only one such vertex-disjoint system (which corresponds to $\sigma = id$)

$$= 1$$