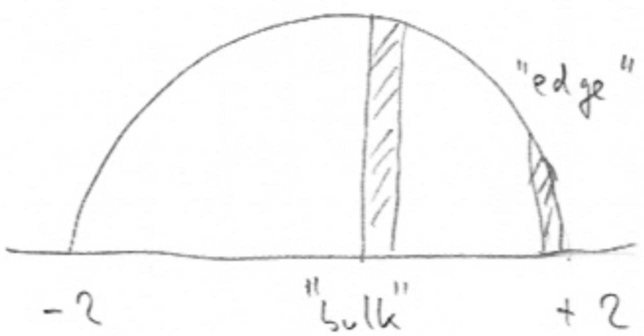
9. Statistics of the largest eigenvalue and Tracy-Widom distribution

Consider GUE(N) or GOE(N).

For large N, the eigenvalue distribution is close to the semicircle



density of semicircle

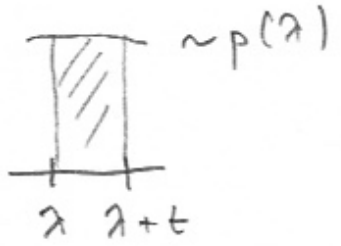
$$p(x) = \frac{1}{2\pi}\sqrt{4-x^2}$$

We will now zoom to a microscopic level and try to understand the behaviour of a single eigenvalue. The behaviour in the bulk and at the edge will be different. We are particularly interested in the largest eigenvalue. Note that at the moment we do not even know whether the largest eigenvalue sticks close to 2 with high probability. Wigner's semicircle law implies that it cannot go much below 2, but it does not prevent it from being very large. We will in particular see that this cannot happen.

9.1 Some heuristics on single eigenvalues:

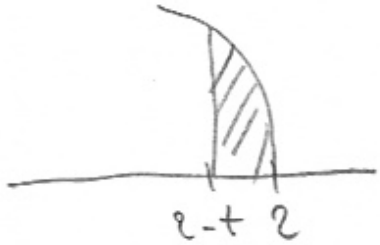
Let us first check heuristically what we expect as typical order of fluctuation of the eigenvalues; for this we assume (without any real justification) that the semicircle predicts the behaviour of eigenvalues down to the microscopic level.

Behaviour in the bulk:



In $[\lambda, \lambda+t]$ there should be $\sim t\, p(\lambda) \cdot N$ eigenvalues. This is of order 1, if we choose $t \sim \frac{1}{N}$; this means one eigenvalue in the bulk has for its position an interval of size $\sim \frac{1}{N}$; so this is a good guess for the order of fluctuation for an eigenvalue in the bulk.

Behaviour at the edge:



In $[2-t, 2]$ there should be roughly

$$N \cdot \int_{2-t}^{2} p(x)\, dx = \frac{1}{2\pi} \int_{2-t}^{2} \sqrt{(2-x)(2+x)}\, dx \cdot N$$

many eigenvalues; to have this of order 1, we should choose t as follows

$$1 \approx \frac{1}{2\pi} \int_{2-t}^{2} \sqrt{(2-x)\underbrace{(2+x)}_{\approx 4 \text{ constant}}}\, dx \cdot N$$

$$\approx \frac{1}{\pi} \underbrace{\int_{2-t}^{2} \sqrt{2-x}\, dx}_{\left[-\frac{2}{3}(2-x)^{3/2}\right]_{2-t}^{2} = \frac{2}{3} t^{3/2}} \cdot N$$

thus: $1 \sim t^{3/2} \cdot N$, i.e. $t \sim N^{-2/3}$

Hence we expect for the largest eigenvalue an interval / fluctuations of size $\sim N^{-2/3}$; very optimistically, we might expect

$\lambda_{max} \approx 2 + N^{-2/3} \cdot X$ where X has an N-independent distribution

9.2. The miracle: This heuristics is indeed true (at least its implication) and one has that the following limit

$$F_\beta(x) := \lim_{N \to \infty} P\{ N^{+2/3}(\lambda_{max} - 2) \leq x \}$$

exists. It is called Tracy-Widom distribution.

9.3 Remarks: 1) Note the parameter β!

This corresponds to

GOE $\beta = 1$

GUE $\beta = 2$

GSE $\beta = 4$

It turns out that the statistics of the largest eigenvalue is different for real, complex, quaternionic Gaussian random matrices. The behaviour on the microscopic level is more sensitive to the underlying symmetry than the macroscopic behaviour. (We get the semicircle for all three ensembles GOE, GUE, GSE.)

[In models in physics the choice of β corresponds often to underlying physical symmetries; e.g. GOE is used to describe systems which have a time-reversal symmetry.]

2) On the other hand, when β is fixed somehow, there seems to be a huge universality class for Tracy-Widom. F_β shows up as limiting fluctuations for:

i) GUE (Tracy-Widom ~ 1993)

ii) more general Wigner matrices (Soshnikov 1999)

iii) general unitarily invariant random matrix ensembles (Deift + Co, ~ 94 - 2000

iv) length of the longest increasing subsequence of random permutation
(Baik, Deift, Johansson 1999
Okounkov 2000)

v) fluctuations of arctic circle for Aztec diamond
(Johansson 2005)

vi) various growth processes
ASEP ("asymmetric simple exclusion process")

TASEP ("totally asymmetric ... ")

3) There is still no uniform explanation for this universality. The feeling is that Tracy-Widom is somehow the analogue of the normal distribution, for a kind of central limit theorem where independence is replaced by some kind of dependence. But no one can make this precise at the moment!

4) Proving Tracy - Widom for GUE is out of reach for us, but we will give some ideas; in particular, we try to derive rigorous estimates which show that our $N^{-2/3}$ - heuristics is the right order (and that the largest eigenvalue converges indeed to 2)

9.4. How to estimate $P(\lambda_{max} \geq 2+\varepsilon)$?

We want to derive an estimate, in the GUE case, for the probability

$$P(\lambda_{max} \geq 2+\varepsilon)$$

which is compatible with our heuristics that $\varepsilon = N^{-2/3} \cdot x$

We will refine our moment method for this.

We have for all $k \in \mathbb{N}$: A_N is our normalized GUE(

$$P\{\lambda_{max} \geq 2+\varepsilon\} = P\{\lambda_{max}^{2k} \geq (2+\varepsilon)^{2k}\}$$

$$\leq P\{\underbrace{\sum_{j=1}^{N} \lambda_j^{2k}}_{= N \cdot tr(A_N^{2k})} \geq (2+\varepsilon)^{2k}\}$$

$$= P\{tr(A_N^{2k}) \geq \frac{(2+\varepsilon)^{2k}}{N}\}$$

$$\overset{\text{Markov}}{\leq} \frac{N}{(2+\varepsilon)^{2k}} E[tr(A_N^{2k})]$$