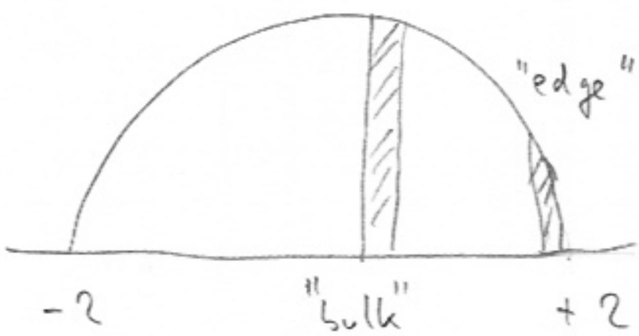
9. Statistics of the largest eigenvalue and Tracy - Widom distribution

Consider GUE(N) or GOE(N).

For large N, the eigenvalue distribution is close to the semicircle



density of semicircle

$$p(x) = \frac{1}{2\pi}\sqrt{4-x^2}$$

We will now zoom to a microscopic level and try to understand the behaviour of a single eigenvalue. The behaviour in the bulk and at the edge will be different. We are particularly interested in the largest eigenvalue. Note that at the moment we do not even know whether the largest eigenvalue sticks close to 2 with high probability. Wigner's semicircle law implies that it cannot go much below 2, but it does not prevent it from being very large. We will in particular see that this cannot happen.

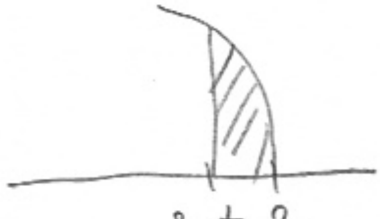
9.1 Some heuristics on single eigenvalues:

Let us first check heuristically what we expect as typical order of fluctuation of the eigenvalues; for this we assume (without any real justification) that the semicircle predicts the behaviour of eigenvalues down to the microscopic level.

Behaviour in the bulk:

In $[\lambda, \lambda+t]$ there should be $\sim t\, p(\lambda)\cdot N$ eigenvalues. This is of order 1, if we choose $t \sim \frac{1}{N}$; this means one eigenvalue in the bulk has for its position an interval of size $\sim \frac{1}{N}$; so this is a good guess for the order of fluctuation for an eigenvalue in the bulk.

Behaviour at the edge:



In $[2-t, 2]$ there should be roughly

$$N\cdot \int_{2-t}^{2} p(x)\,dx = \frac{1}{2\pi}\int_{2-t}^{2} \sqrt{(2-x)(2+x)}\,dx\cdot N$$

many eigenvalues; to have this of order 1, we should choose t as follows

$$1 \approx \frac{1}{2\pi}\int_{2-t}^{2} \sqrt{(2-x)\underbrace{(2+x)}_{\approx 4 \text{ constant}}}\, dx \cdot N$$

$$\approx \frac{1}{\pi}\underbrace{\int_{2-t}^{2} \sqrt{2-x}\, dx}_{\left[-\frac{2}{3}(2-x)^{3/2}\right]_{2-t}^{2} = \frac{2}{3}t^{3/2}} \cdot N$$

thus: $1 \sim t^{3/2} \cdot N$, i.e. $t \sim N^{-2/3}$

Hence we expect for the largest eigenvalue an interval / fluctuations of size $\sim N^{-2/3}$; very optimistically, we might expect

$\lambda_{max} \approx 2 + N^{-2/3} \cdot X$ where X has an N-independent distribution

<u>9.2. The miracle</u>: This heuristics is indeed true (at least its implication) and one has that the following limit

$$F_\beta(x) := \lim_{N\to\infty} P\{N^{+2/3}(\lambda_{max} - 2) \le x\}$$

exists. It is called <u>Tracy-Widom distribution</u>.

9.3 Remarks: 1) Note the parameter β!

This corresponds to

GOE	$\beta = 1$
GUE	$\beta = 2$
GSE	$\beta = 4$

It turns out that the statistics of the largest eigenvalue is different for real, complex, quaternionic Gaussian random matrices. The behaviour on the microscopic level is more sensitive to the underlying symmetry than the macroscopic behaviour. (We get the semicircle for all three ensembles GOE, GUE, GSE.)

[In models in physics the choice of β corresponds often to underlying physical symmetries; e.g. GOE is used to describe systems which have a time-reversal symmetry.]

2) On the other hand, when β is fixed somehow, there seems to be a huge universality class for Tracy-Widom. F_β shows up as limiting fluctuations for:

i) GUE (Tracy-Widom ~ 1993)

ii) more general Wigner matrices (Soshnikov 1999)

iii) general unitarily invariant random matrix ensembles (Deift + Co, ~ 94-2000

iv) length of the longest increasing subsequence of random permutation (Baik, Deift, Johansson 1999
Okounkov 2000)

v) fluctuations of arctic circle for Aztec diamond
(Johansson 2005)

vi) various growth processes
ASEP ("asymmetric simple exclusion process")

TASEP ("totally asymmetric ... ")

3) There is still no uniform explanation for this universality. The feeling is that Tracy-Widom is somehow the analogue of the normal distribution, for a kind of central limit theorem where independence is replaced by some kind of dependence. But no one can make this precise at the moment!

4) Proving Tracy-Widom for GUE is out of reach for us, but we will give some ideas; in particular, we try to derive rigorous estimates which show that our $N^{-2/3}$-heuristics is the right order (and that the largest eigenvalue converges indeed to 2)

9.4. How to estimate $P(\lambda_{max} \geq 2+\varepsilon)$?

We want to derive an estimate, in the GUE case, for the probability

$$P(\lambda_{max} \geq 2+\varepsilon)$$

which is compatible with our heuristics that $\varepsilon = N^{-2/3} \cdot x$

We will refine our moment method for this.

We have for all $k \in \mathbb{N}$: A_N is our normalized GUE(N)

$$P\{\lambda_{max} \geq 2+\varepsilon\} = P\{\lambda_{max}^{2k} \geq (2+\varepsilon)^{2k}\}$$

$$\leq P\{\underbrace{\sum_{j=1}^{N} \lambda_j^{2k}}_{= N \cdot tr(A_N^{2k})} \geq (2+\varepsilon)^{2k}\}$$

$$= P\{tr(A_N^{2k}) \geq \frac{(2+\varepsilon)^{2k}}{N}\}$$

$$\overset{\text{Markov}}{\leq} \frac{N}{(2+\varepsilon)^{2k}} E[tr(A_N^{2k})]$$

In 2.13 we calculated the expectation in terms of a genus expansion as

$$E[\operatorname{tr}(A_N^{2k})] = \sum_{\pi \in \mathcal{P}_2(2k)} N^{\#(\gamma\pi)-k-1}$$

$$= \sum_{g \geq 0} \varepsilon_g(k) N^{-2g} \qquad \text{(see 2.15(3))}$$

where

$$\varepsilon_g(k) = \#\{\pi \in \mathcal{P}_2(2k) \mid \pi \text{ has genus } g\}$$

i.e. $\#(\gamma\pi)-k-1 = -2g$

The inequality

$$P\{\lambda_{max} \geq 2+\varepsilon\} \leq \frac{N}{(2+\varepsilon)^{2k}} E[\operatorname{tr}(A_N^{2k})]$$

is useless if k is fixed for $N \to \infty$, because then the right hand side goes to ∞. Hence we have to scale k also with N (we will use $k \sim N^{2/3}$), but then the subleading terms in $E[\operatorname{tr}(A_N^{2k})]$ become important. Up to now we only know $\varepsilon_0(k) = C_k$, but now we need also some info on the other $\varepsilon_g(k)$. This is provided by a theorem of Harer and Zagier.

9.5. Theorem (Harer - Zagier 1986): Let us define b_k by

$$\sum_{g\geq 0} \varepsilon_g(k)\cdot N^{-2g} = C_k \cdot b_k$$

where C_k are the Catalan numbers.
[Note that b_k depends on N.]
Then we have the recursion

$$b_{k+1} = b_k + \frac{k(k+1)}{4N^2} b_{k-1} \qquad \forall k \geq 2$$

9.6. Example: We will prove this later; for now let us only check a few small examples.
We know (see 2.14.):

$$C_1 \cdot b_1 = E[\mathrm{tr}(A_N^2)] = 1 \qquad \Rightarrow b_1 = 1$$

$$C_2 \cdot b_2 = E[\mathrm{tr}(A_N^4)] = 2 + \frac{1}{N^2}$$

$$\Rightarrow b_2 = 1 + \frac{1}{2N^2}$$

$$C_3 \cdot b_3 = E[\mathrm{tr}(A_N^6)] = 5 + 10\frac{1}{N^2}$$

$$\Rightarrow b_3 = 1 + \frac{2}{N^2}$$

$$C_4 \cdot b_4 = E[\mathrm{tr}(A_N^8)] = 14 + 70\frac{1}{N^2} + 21\frac{1}{N^4}$$

$$\Rightarrow b_4 = 1 + \frac{5}{N^2} + \frac{3}{2}\frac{1}{N^4}$$

We check now 9.5 for $k=3$:

$$b_3 + \frac{3\cdot 4}{4N^2} b_2 = 1 + \frac{2}{N^2} + \frac{12}{4N^2}\left(1 + \frac{1}{2N^2}\right)$$

$$= 1 + 5\frac{1}{N^2} + \frac{3}{2}\frac{1}{N^4}$$

$$= b_4$$

9.7 Corollary: For all $N, k \in \mathbb{N}$ we have for an GUE(N) A_N that

$$E[tr(A_N^{2k})] \le C_k \cdot e^{k^3/2N^2}$$

where C_k are the Catalan numbers

Proof: Note that by definition: $b_k > 0$ $\forall k$ and hence, by 9.5., $b_{k+1} > b_k$

Thus we have

$$b_{k+1} = b_k + \frac{k(k+1)}{4N^2} b_{k-1}$$

$$\le b_k \cdot \left(1 + \frac{k(k+1)}{4N^2}\right)$$

$$\le b_k \cdot \left(1 + \frac{k^2}{2N^2}\right)$$

$$\Rightarrow b_k \le \left(1+\frac{(k-1)^2}{2N^2}\right)\left(1+\frac{(k-2)^2}{2N^2}\right)\cdots\left(1+\frac{1^2}{2N^2}\right)$$

$$\le \left(1+\frac{k^2}{2N^2}\right)^k$$

$$\le \left[\exp\left(\frac{k^2}{2N^2}\right)\right]^k$$

$$= \exp\left[\frac{k^3}{2N^2}\right]$$

$\square$

We can now continue our estimate from 9.4:

$$P\{\lambda_{max} \ge 2+\varepsilon\} \le \frac{N}{(2+\varepsilon)^{2k}} \underbrace{E[tr(A_N^{2k})]}_{= C_k \cdot b_k}$$

$$\overset{9.7}{\le} \frac{N}{(2+\varepsilon)^{2k}} C_k \, e^{k^3/2N^2}$$

For the Catalan numbers we use the estimate (check via Stirling formula!)

$$C_k \le \frac{4^k}{k^{3/2}\sqrt{\pi}} \le \frac{4^k}{k^{3/2}}$$

Thus

$$P\{\lambda_{max} \ge 2+\varepsilon\} \le \frac{N}{(2+\varepsilon)^{2k}} \cdot \frac{4^k}{k^{3/2}} e^{+k^3/2N^2} \qquad (*)$$

Let us first fix $\varepsilon > 0$ and choose

$k_N = \lfloor N^{2/3} \rfloor$ (smallest integer $\geq N^{2/3}$);

then

$$\frac{N}{k_N^{3/2}} \to 1 \quad \text{and} \quad \frac{k_N^3}{2N^2} \to \frac{1}{2}$$

hence

$$\overline{\lim_{N\to\infty}} P\{\lambda_{max} \geq 2+\varepsilon\} \leq \underbrace{\lim_{N\to\infty} \left(\frac{2}{2+\varepsilon}\right)^{2k_N}}_{=0} \cdot 1 \cdot e^{1/2}$$

Hence, for each $\varepsilon > 0$,

$$\lim_{N\to\infty} P\{\lambda_{max} \geq 2+\varepsilon\} = 0$$

and thus λ_{max} converges in probability to 2.

<u>9.8. Corollary</u>: For a GUE(N) A_N we have that its largest eigenvalue converges in probability, and also almost surely, to 2,

$$\lambda_{max}(A_N) \to 2 \qquad \text{almost surely}$$

Proof: For the almost sure version one has to use Borel-Cantelli and the fact that

$$\sum_N \left(\frac{2}{2+\varepsilon}\right)^{2k_N} < \infty$$ ☐

9.9. Remark: (*)
Our estimate gives now also some information about the fluctuation of λ_{max} around 2 if we choose also ε depending on N.
So let us use in (*) now
$k_N = \lfloor N^{2/3} \rfloor r$ and $\varepsilon_N = N^{-2/3} \cdot t$; then

$$\frac{N}{k_N^{3/2}} \to \frac{1}{r^{3/2}} \quad , \quad \frac{k_N^3}{2N^2} \to \frac{r^3}{2}$$

and

$$\frac{4^{k_N}}{(2+\varepsilon_N)^{2k_N}} = \left(\frac{1}{1+\frac{1}{N^{2/3}}\cdot\frac{t}{2}}\right)^{2\cdot \lfloor N^{2/3} \rfloor \cdot r}$$

$$\to e^{-r\cdot t}$$

and thus

$$\overline{\lim_{N\to\infty}}\, P\{\lambda_{max} \geq 2 + tN^{-2/3}\} \leq e^{-r\cdot t}\,\frac{1}{r^{3/2}}\,e^{+r^3/2}$$

for arbitrary $r > 0$

We optimize this now by choosing $r = \sqrt{t}$ (for $t \geq 0$) and get then

$$e^{-t^{1/2}\cdot t} \frac{1}{(t^{1/2})^{3/2}} \cdot e^{(t^{1/2})^3 \cdot \frac{1}{2}} = e^{-\frac{1}{2}t^{3/2}} \cdot t^{-3/4}$$

hence

$$\overline{\lim_{N\to\infty}} P\{\lambda_{max} \geq 2 + tN^{-2/3}\} \leq t^{-3/4} e^{-\frac{1}{2}t^{3/2}}$$

This estimate is of the right order.

9.10 Remark: For a determination of the Tracy-Widom fluctuations in the limit $N \to \infty$ one has to use the analytic descripti[on] of the GUE-joint density.

Recall from 7.20 that the joint density of the unordered eigenvalues of an unnormalized GUE(N) is given by

$$p(\mu_1, \dots \mu_N) = \frac{1}{N!} \det(K_N(\mu_i, \mu_j))_{i,j=}^{N}$$

where K_N is the Hermite kernel

$$K_N(x,y) = \sum_{k=0}^{N-1} \Psi_k(x)\Psi_k(y)$$

where Ψ_k are the Hermite functions.

Because K_N is a reproducing kernel, we can integrate out some of the eigenvalues and get a density of the same form. If we are integrating out all but r eigenvalues we get by 7.18.

$$\int \dots \int p(\mu_1, \dots, \mu_N)\, d\mu_{r+1}\, d\mu_{r+2} \cdots d\mu_N =$$

$$= \frac{1}{N!}\, 1 \cdot 2 \cdots (N-r) \quad \det\left(K_N(\mu_i, \mu_j)\right)_{i,j=1}^{r}$$

$$= \frac{(N-r)!}{N!} \det\left(K_N(\mu_i, \mu_j)\right)_{i,j=1}^{r} =: p_N(\mu_1, \dots \mu_r)$$

Consider now

$$P\{\mu_{max}^{(N)} \le t\} = P\{\text{no eigenvalue in } (t, \infty)\}$$

$$= 1 - \underbrace{P\{\exists \text{ eigenvalue in } (t,\infty)\}}$$

$$N\, P\{\mu_1 \text{ in } (t,\infty)\}$$

$$- \binom{N}{2} P\{\mu_1, \mu_2 \text{ in } (t,\infty)\}$$

$$+ \binom{N}{3} P\{\mu_1, \mu_2, \mu_3 \text{ in } (t,\infty)\}$$

$$- + \dots$$

$$= 1 + \sum_{r=1}^{N} (-1)^r \binom{N}{r} \underbrace{\int_t^\infty \dots \int_t^\infty p_N(\mu_1, \dots, \mu_r)\, d\mu_1 \dots d\mu_r}_{\frac{1}{r!} \int_t^\infty \dots \int_t^\infty \det\left(K_N(\mu_i, \mu_j)\right)_{i,j=1}^{r} d\mu_1 \dots d\mu_r}$$

Does this have a limit for $N \to \infty$?

Note: p is distribution for GUE(N) without normalisation, i.e. $\mu_{max}^{(N)} \approx 2\sqrt{N}$; more precisely, we expect fluctuations

$$\mu_{max}^{(N)} \approx \sqrt{N}\,(2 + t N^{-2/3})$$
$$= 2\sqrt{N} + t N^{-1/6}$$

and we put

$$\tilde{K}_N(x,y) := N^{-1/6} K_N(2\sqrt{N} + x N^{-1/6}, 2\sqrt{N} + y N^{-1/6})$$

so that we have

$$P\{ N^{2/3} \left(\frac{\mu_{max}^{(N)}}{\sqrt{N}} - 2 \right) \leq t \} =$$

$$= \sum_{k=0}^{N} \frac{(-1)^r}{r!} \int_t^\infty \dots \int_t^\infty \det(\tilde{K}_N(x_i, x_j))_{i,j=1}^{r} \, dx_1 \dots dx_r$$

We expect that the limit

$$F_2(t) := \lim_{N\to\infty} P\{ N^{2/3} \left(\frac{\mu_{max}}{\sqrt{N}} - 2 \right) \leq t \}$$

exists; for this we need the limit

$$\lim_{N\to\infty} \tilde{K}_N(x,y)$$

Recall that

$$K_N(x,y) = \sum_{k=0}^{N-1} \Psi_k(x)\,\Psi_k(y)$$

As this involves Ψ_k for all $k = 0, \dots, N-1$ this is not amenable to taking a limit $N \to \infty$. However, by the Christoffel-Darboux identity (see Assignment 9, Exercise 3) for the Hermite functions

$$\sum_{k=0}^{n-1} \frac{H_k(x)\,H_k(y)}{k!} = \frac{H_n(x)\,H_{n-1}(y) - H_{n-1}(x)\,H_n(y)}{(n-1)!\,(x-y)}$$

and with (see 7.12)

$$\Psi_k(x) = (2\pi)^{-1/4}\,(k!)^{-1/2}\,e^{-\frac{1}{4}x^2}\,H_k(x)$$

we can rewrite K_N as

$$K_N(x,y) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N-1} \frac{1}{k!}\, e^{-\frac{1}{4}(x^2+y^2)}\, H_k(x)\,H_k(y)$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{4}(x^2+y^2)}\, \frac{H_N(x)\,H_{N-1}(y) - H_{N-1}(x)\,H_N(y)}{(N-1)!\,(x-y)}$$

$$= \sqrt{N}\, \frac{\Psi_N(x)\,\Psi_{N-1}(y) - \Psi_{N-1}(x)\,\Psi_N(y)}{(x-y)}$$

Note that φ_N satisfies the differential equation (-> assignment)

$$\varphi_N'(x) = -\frac{x}{2}\varphi_N(x) + \sqrt{N}\,\varphi_{N-1}(x),$$

and thus $K_N(x,y) =$

$$= \frac{\varphi_N(x)\left[\varphi_N'(y) + \frac{y}{2}\varphi_N(y)\right] - \left[\varphi_N'(x) + \frac{x}{2}\varphi_N(x)\right]\varphi_N(}{x-y}$$

$$= \frac{\varphi_N(x)\varphi_N'(y) - \varphi_N(y)\varphi_N'(x)}{x-y} - \frac{1}{2}\varphi_N(x)\varphi_N(y)$$

Now put

$$\tilde{\varphi}_N(x) := N^{1/12}\varphi_N(2\sqrt{N} + xN^{-1/6}), \quad \text{thus}$$

$$\tilde{\varphi}_N'(x) = N^{1/12}\cdot N^{-1/6}\,\varphi_N'(\dots)$$

$$= N^{-1/12}\varphi_N'(2\sqrt{N} + xN^{-1/6}),$$

then we have

$$\tilde{K}_N(x,y) = N^{-1/6}\,\frac{\tilde{\varphi}_N(x)N^{-1/12}\cdot\tilde{\varphi}_N'(y)N^{+1/2} - \dots}{N^{-1/6}(x-y)}$$

$$- N^{-1/6}\,\frac{1}{2}\,\tilde{\varphi}_N(x)N^{-1/12}\cdot\tilde{\varphi}_N(y)N^{-1/12}$$

$$= \frac{\tilde{\varphi}_N(x)\tilde{\varphi}_N'(y) - \tilde{\varphi}_N'(x)\tilde{\varphi}_N(y)}{x-y} - \frac{1}{2N^{1/3}}\tilde{\varphi}_N(x)\tilde{\varphi}_N$$

One can now show (by a non-trivial steepest descent method) that $\tilde{\varphi}_N(x)$ converges to a limit; let us call this

$$Ai(x) := \lim_{N\to\infty} \tilde{\varphi}_N(x) \qquad \text{Airy function}$$

the convergence is so strong that also

$$Ai'(x) = \lim_{N\to\infty} \tilde{\varphi}_N'(x)$$

and hence

$$\lim \tilde{k}_N(x,y) = \frac{Ai(x)\,Ai'(y) - Ai'(x)\,Ai(y)}{x-y}$$

$$=: Ai(x,y)$$

Airy kernel

Let us try to characterize this limit fct Ai.

The Hermite functions φ_N satisfy the differential equation:

$$\varphi_N''(x) + \left(N + \frac{1}{2} - \frac{x^2}{4}\right)\varphi_N(x) = 0;$$

for the $\tilde{\varphi}_N$ we have

$$\tilde{\varphi}_N'(x) = N^{-1/12}\,\varphi_N'(2\sqrt{N} + xN^{-1/6})$$

$$\tilde{\varphi}_N''(x) = N^{-1/4}\,\varphi_N''(2\sqrt{N} + xN^{-1/6})$$

and thus

$$\tilde{\Psi}_N''(x) = -N^{-1/4}\left[N+\frac{1}{2}-\frac{(2\sqrt{N}+xN^{-1/6})^2}{4}\right]\underbrace{\Psi_N(2\sqrt{N}+xN^{-1/6}}_{N^{-1/12}\cdot\tilde{\Psi}_N(x}$$

$$= -N^{-1/3}\left[\cancel{N}+\frac{1}{2}-\frac{4\cancel{N}+4xN^{1/3}+x^2N^{-1/3}}{4}\right]\tilde{\Psi}_N(x)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad x \qquad 0$$

$$\simeq -x\,\tilde{\Psi}_N(x)$$

hence we expect that Ai should satisfy

$$Ai''(x) - x\,Ai(x) = 0$$

<u>9.11 Def</u>: The <u>Airy function</u>

$$Ai : \mathbb{R} \to \mathbb{R}$$

is a solution to the Airy ODE

$$u''(x) = x\,u(x)$$

determined by the following asymptotics as $x \to \infty$

$$Ai(x) \sim \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-2/3\,x^{3/2}}$$

The <u>Airy kernel</u> is defined by

$$A(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$$

9.12. Theorem: The random variable $N^{2/3}(\lambda_{max}-2)$ of the normalized GUE has a limit distribution as $N \to \infty$; its limiting cumulative distribution function is

$$F_2(t) := \lim_{N\to\infty} P\{N^{2/3}(\lambda_{max}-2) \leq t\}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \int_t^{\infty} \dots \int_t^{\infty} \det(A(x_i, x_j))_{i,j=1}^{r} \, dx_1 \dots dx_r$$

9.13. Theorem (Tracy + Widom 1994): The distribution function F_2 satisfies

$$F_2(t) = \exp\left\{ - \int_t^{\infty} (x-t)\, q(x)^2 \, dx \right.$$

where q is a solution of the Painlevé II equation

$$q''(x) - x\,q(x) + 2\,q(x)^3 = 0$$

and

$$q(x) \sim Ai(x) \quad \text{as } x \to \infty .$$