UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Exercices to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 1
submission: Tuesday, October 24 2017, 2 pm
postbox of Vincent Preiß (basement of building E2.5)

Exercise 1 (10 points)**.** Let $f : X \to Y$ be a mapping netween topological spaces. Show, that f is continuous (in the sense of definition 1.3 of the lecture) if and only if, for every net $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ with $x_\lambda \to x$ we already have $f(x_\lambda) \to f(x)$. This proves remark 1.9(d) of the lecture.

product topology. If X and Y are topological spaces, then the *product topology* on $X\times Y$ is generated by the set $\{U \times V \mid U \subseteq X, V \subseteq Y \text{ both open}\}$. Hence open sets in $X \times Y$ are arbitrary unions of such sets $U \times V$. Accordingly, $N \subseteq X \times Y$ is a neighbourhood of $(x, y) \in X \times Y$ if there are open sets $U \subseteq X, V \subseteq Y$ with $x \in U, y \in V, U \times V \subseteq N$. So a net $(x_\lambda, y_\lambda)_{\lambda\in\Lambda}$ converges to (x, y) if and only if $x_\lambda \to x$ and $y_\lambda \to y$. It follows from exercise 1, that $f : X \times Y \to Z$ is continuous if and only if for all sequences $x_\lambda \to x$, $y_\lambda \to y$ we have $f(x_\lambda, y_\lambda) \to f(x, y)$.

Exercise 2 (10 points)**.** let X be a topological space. Show, that the following are equivalent.

(a) Each two points in X have disjoint neighbourhoods.

(b) Every point in X is the intersection of its closed neighbourhoods.

(c) The diagonal in $X \times X$ is closed.

(d) No net in X converges against two different points.

If one of the above (and so all of them) are fulfilled, the space X is called a *Hausdorff space* (remark: Metric spaces are Hausdorff spaces.)

please turn the page

Exercise 3 (10 points)**.** Let the usual topology on $\mathbb{R}^2$ be extended to a topology on $\mathcal{R}^2 := \mathbb{R}^2 \cup \{\infty\}$ by defining the sets U_α for $\alpha \in A$ as neighbourhoods of ∞ (they form a neighbourhood system of ∞). Here

$$\begin{aligned} U_\alpha &:= \{(x,y) \in \mathbb{R}^2 \mid x > c_\alpha, y > f_\alpha(x)\} \cup \{\infty\} \\ A &:= \{(c_\alpha, f_\alpha) \mid c_\alpha \in \mathbb{R}, f_\alpha : \mathbb{R} \to \mathbb{R} \text{ continuous, strictly increasing, unbounded}\} \end{aligned}$$

(a) Prove, that there is no sequence in $\mathbb{R}^2$ converging to ∞.

(b) Construct a net in $\mathbb{R}^2$, converging against ∞.

Exercise 4 (10 points)**.** Let Y be a complete metric space and $X \subseteq Y$ a subset. Prove, that the completion $\hat{X}$ is isometrically isomorphic to the closure $\bar{X}$ of X. (This is a part of Theorem 1.12).