UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 2
submission: Monday, October 30 2017, 2 pm
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Exercise 1 (10 points)**.** We consider the Banach space $X = \mathcal{C}[0,1]$ of continuous, complex-valued functions on $[0,1]$. Give each one example of continuous, linear mappings $T : X \to X$ which are

(a) injective but not surjective and

(b) surjective but not injective.

(By the dimension formula, this is not possible in the finite-dimensional situation.)

Exercise 2 (20 points)**.** We define the following spaces of sequences:

$$\begin{aligned}
f &:= \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{C}, \exists N \in \mathbb{N}\ \forall n \geq N : a_n = 0\} && \\
c_0 &:= \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{C}, \lim_{n\to\infty} a_n = 0\}, && \|(a_n)\|_\infty := \sup_{n\in\mathbb{N}} |a_n| \\
\ell^1 &:= \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{C}, \sum_{n=0}^{\infty} |a_n| < \infty\}, && \|(a_n)\|_1 := \sum_{n=0}^{\infty} |a_n| \\
\ell^\infty &:= \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{C}, \sup_{n\in\mathbb{N}} |a_n| < \infty\}, && \|(a_n)\|_\infty := \sup_{n\in\mathbb{N}} |a_n|
\end{aligned}$$

Note, that $(c_0, \|\cdot\|_\infty)$, $(\ell^1, \|\cdot\|_1)$ and $(\ell^\infty, \|\cdot\|_\infty)$ are complete.

(a) Prove with the help of the bilinear mapping $\sigma : (a_n) \times (b_n) \mapsto \sum_{n=1}^{\infty} a_n b_n$ that for the dual spaces it holds $(c_0, \|\cdot\|_\infty)' \cong (\ell^1, \|\cdot\|_1)$ and $(\ell^1, \|\cdot\|_1)' \cong (\ell^\infty, \|\cdot\|_\infty)$.

(b) Prove the completions $(f, \|\cdot\|_1)\hat{}\cong (\ell^1, \|\cdot\|_1)$ and $(f, \|\cdot\|_\infty)\hat{}\cong (c_0, \|\cdot\|_\infty)$.

(c) Consider the mappings $(f, \|\cdot\|_1) \to (f, \|\cdot\|_\infty)$ and $(f, \|\cdot\|_\infty) \to (f, \|\cdot\|_1)$, defined by the identity mapping. Investigate the continuity of these two mappings.

Exercise 3 (10 points)**.** Let V be a $\mathbb{C}$-vector space together with a metric $d : V \times V \to \mathbb{R}$. Formulate conditions that are equivalent to the fact, that (V, d) is already a normed space (i.e. there exists a norm $\|\cdot\|$ on V such that d is induced by $\|\cdot\|$). Prove your claim.