UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 3
submission: Tuesday, November 14 2017, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (10 points)**.** Let X be a locally-convex vector space with countably many semi-norms $\{p_0, p_1, \ldots\}$, such that $p_n(x) = 0$ for all $n \in \mathbb{N}_0$ already implies $x = 0$. Show that

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)}$$

defines a metric on X, inducing the same topology as the semi-norm system. Show that in the case $X = \mathcal{C}^\infty[0, 1]$ with $p_n(f) := \|f^{(n)}\|_\infty$ (like in Example 1.35(a) of the lecture) the vector space X is complete. (This metric does not come from a norm.)
Hint: From the Analysis course you (should) know, that the pointwise limit f of a sequence (f_n) of continuous, differentiable functions is again differentiable if the sequence (f_n') converges to the function g uniformly. In this case we have $f' = g$.

Exercise 2 (10 points)**.** A metric space (X, d) is called *separable*, if there is a countable subset $A \subseteq X$, which is dense in X (Convince yourself that $\mathbb{R}$ and $\mathbb{C}$ are separable.). Show that $(c_0, \|\cdot\|_\infty)$ is separable, but $(\ell^\infty, \|\cdot\|_\infty)$ is not.

Exercise 3 (10 points)**.** Show, that there exist discontinuous, linear functionals on every infinite-dimensional normed space V.

Exercise 4 (10 points)**.** Let E be a normed space and E' its dual. Show that the norm on E is given by

$$\|x\| = \sup\left\{|f(x)| \;\middle|\; f \in E', \|f\| \leq 1\right\}.$$

Furthermore, the supremum is a maximum, i.e. for $x \in E$ there is a $f \in E'$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

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Exercise 5 (10 points)**.** We consider the Banach space $\ell^\infty_\mathbb{R}$ of real-valued bounded sequences together with the supremum-norm. Show that $q : \ell^\infty_\mathbb{R} \to \mathbb{R}$, given by the limes superior $q((a_n)) = \text{lim sup}_{n\to\infty} a_n$, is a sublinear functional. Deduce that there is a bounded, linear functional $L : \ell^\infty_\mathbb{R} \to \mathbb{R}$ with

$$\text{lim inf}_{n\to\infty} a_n \leq L((a_n)) \leq \text{lim sup}_{n\to\infty} a_n.$$

Additionally show:

- $L(1) = 1$, where $1 \in \ell^\infty_\mathbb{R}$ is the constant sequence $(1)_{n\in\mathbb{N}}$.
- $L((a_n)) \geq 0$ if all $a_n \geq 0$.
- $\|L\| = 1$.
- If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, it holds $L((a_n)) = \lim_{n\to\infty} a_n$.

In some sense, this enables us to define a "limit" for bounded sequences. Restricting L to convergent sequences gives back the usual limit of sequences.

Exercise 6 (10 points)**.** Show with the help of an example, that, in general, the convex hull of a compact subset of a normed space is not closed. For this, consider for example a sequence in $(c_0, \|\cdot\|_\infty)$ converging to the the constant sequence with value 0.