UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
Prof. Dr. Moritz Weber
Stefan Jung

Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 4
submission: Tuesday, November 21 2017, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (20 points)**.** For $1 \leq p \leq \infty$ we consider the Banach spaces

$$\ell^p := \left\{(a_n)_{n\in\mathbb{N}} \middle| a_n \in \mathbb{C}, \|(a_n)\|_p < \infty\right\}$$

$$\text{where} \quad \|(a_n)\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} \qquad 1 \leq p < \infty$$

$$\|(a_n)\|_\infty := \sup_{n\in\mathbb{N}}|a_n| \qquad p = \infty$$

(a) Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual space of ℓ^p is isometrically isomorphic to ℓ^q.

Hint: Use Hölder's inequality $\|(a_n)(b_n)\|_1 \leq \|(a_n)\|_p\|(b_n)\|_q$, where the product $(a_n)_{n\in\mathbb{N}}(b_n)_{n\in\mathbb{N}}$ is defined by $(a_nb_n)_{n\in\mathbb{N}}$.

(b) Let $1 \leq p < q \leq \infty$. Show that $\ell^p \subsetneq \ell^q$ and $\|(a_n)\|_q \leq \|(a_n)\|_p$.

(c) Let $(a_n) \in \ell^p$ for all $1 \leq p < \infty$. Show that $\lim_{q\to\infty}\|(a_n)\|_q = \|(a_n)\|_\infty$.

(d) What's going wrong if we try to make ℓ^p for $0 < p < 1$ a normed space?

Exercise 2 (10 points)**.** Show, that the unit ball of the Banach space $(c_0, \|\cdot\|_\infty)$, $\{(a_n)\in c_0 \mid \|(a_n)\|_\infty \leq 1\}$, has no extreme points.

Exercise 3 (10 points)**.** We consider the space $(c, \|\cdot\|_\infty)$, where:

$$c := \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{C}, \lim_{n\to\infty} a_n \text{ exists}\}, \qquad \|(a_n)\|_\infty := \sup_{n\in\mathbb{N}}|a_n|$$

Show that the dual space of c is ℓ^1. (So $c_0 \subsetneq c$ and $c_o' = c'$.) For this, show first, that the mapping $F : c \to \mathbb{C}$, $(a_n) \mapsto \lim_{n\to\infty} a_n$ is continuous and linear.

please turn the page

Additional exercise [10 extra points]:

(i) Let $T : X \to Y$ be a linear mapping between vector spaces. Show that the pre-image $T^{-1}(A)$ of convex sets $A \subseteq Y$ is again convex.

For $0<p<1$ we define now the vector space

$$L^p := \{f : [0,1] \to \mathbb{C} \mid f \text{ Borel-measurable and } \int_{[0,1]} |f(x)|^p d\lambda(x) < \infty\},$$

where λ is the usual Borel-measure on $\mathbb{R}$ restricted to $[0,1]$ and we identify as usual functions f_1 and f_2 if they only differ on zero-sets.

(ii) Show that $d(f,g) := \int_{[0,1]} |f(x) - g(x)|^p d\lambda(x)$ defines a metric on L^p.

(iii) Use for given $n \in \mathbb{N}$ and $0 = x_0 < x_1 < \ldots < x_n = 1$ the convex combination

$$f = \frac{1}{n}\left(nf \cdot \mathbb{1}_{[x_0,x_1]} + nf \cdot \mathbb{1}_{[x_1,x_2]} + \ldots + nf \cdot \mathbb{1}_{[x_{n-1},x_n]}\right)$$

to show that for every $\varepsilon > 0$ we can write f as a convex combination of elements $g_1, \ldots, g_n \in L^p$ with $d(0, g_i) < \varepsilon$ for all $1 \le i \le n$. This shows, that (L^p, d) is not a locally-convex vector space.

(iv) Deduce that L^p is the only open, convex set of the metric space (L^p, d) containing 0.

(v) Show that $(L^p, d)' = \{0\}$, i.e. there is no non-trivial functional on (L^p, d).

(vi) Prove, that Theorem 2.7 from the lecture does not hold for a general (metric) vector spaces: Not every (continuous) linear functional defined on a closed subspace of a (metric) vector space has a continuous, linear extension to the whole space.