UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 5
submission: Tuesday, November 28 2017, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (10 points)**.** Show that there is a function $g \in (C([0,1]), \|\cdot\|_\infty)$ which is not differentiable in every point of $[0, \frac{1}{2}]$. For this, show that the sets

$$F_n := \left\{ f \in \mathcal{C}([0,1]) \;\middle|\; \exists x \in [0, \frac{1}{2}] : \sup_{0<h<\frac{1}{2}} \frac{|f(x+h) - f(x)|}{h} \leq n \right\}$$

are closed for all $n \in \mathbb{N}$ and contain no open sets.

Exercise 2 (10 points)**.**

(a) Prove Corollary 4.3 of the lecture under usage of Baire's theorem: There is no real or complex Banach space of countably infinite vector space dimension.

(b) Show that the vector space of all polynomials with complex-valued coefficients is not complete with respect to no norm.

Exercise 3 (10 points)**.** Prove Corollary 4.5 of the lecture (under usage of the principle of uniform boundedness): If E is a normed space and $M \subseteq E$ a subset, such that for every $f \in E'$ the set $f(M)$ is bounded, then we already have that the set M is bounded.

Exercise 4 (10 points)**.** We consider f, the space of finite sequences in $\mathbb{C}$ (see sheet 2) as a subspace of ℓ^∞. Let $T : f \to f$ be given by:

$$T(a_1, a_2, a_3, \ldots, a_n, \ldots) := (a_1, 2a_2, 3a_3, \ldots, na_n, \ldots)$$

(a) Show that $f \subseteq \ell^\infty$ is not closed (so not complete).

(b) Show that T is not continuous (but linear).

(c) Show that T is a pointwise limit of continuous, linear mappings $T_n : f \to f$.

So the Banach-Steinhaus theorem (Cor. 4.6 of the lecture) is wrong if X is no Banach space.