UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 6
submission: Tuesday, December 5 2017, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (10 points)**.** Consider the Hilbert space $\mathbb{R}^2$ with $\langle x, y\rangle = \sum_{i=1}^{2} x_i y_i$.

(a) Describe all unit vectors (i.e. vectors with norm 1) with the help of the functions sin and cos.

(b) Describe all unit vectors, that are orthogonal to a given unit vector $x = (x_1, x_2) \in \mathbb{R}^2$.

(c) Show that $\frac{\langle x,y\rangle}{\|x\|\|y\|} = \cos\varphi$, where φ is the angle between the vectors x und y.

Exercise 2 (10 points)**.** (a) Let H be a pre-Hilbert space. Show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \text{for all } x, y \in H.$$

(b) Use (a) to show that the supremum norm on $\mathcal{C}[0,1]$ does not come from a scalar product (i.e. the supremum norm cannot be written as $\|f\| = \sqrt{\langle f, f\rangle}$).

Exercise 3 (10 points)**.** Let H be a Hilbert space and $F \subseteq H$ a linear subspace. Show, that $(F^{\perp})^{\perp} = \overline{F}$.

Exercise 4 (10 points)**.** Let H be a separable Hilbert space.

(a) Let $N \in \mathbb{N} \cup \{\infty\}$ and $x_1, \ldots, x_N$ linearly independent elements in H. Show that for all $i \in \mathbb{N}$ with $1 \leq i \leq N$ there is a ONS $e_1, \ldots, e_N$ in H such that the linear span of $x_1, \ldots, x_i$ is equal to that of $e_1, \ldots, e_i$.

Hint: Use the Gram-Schmidt orthogonalization process from the lecture.

(b) Deduce that H has a countable ONB.

(c) Conversely, show (in a more explicit way than in the corresponding remark in the lecture): If H has a countable ONB, then H is separable.