UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 7
submission: Tuesday, December 12 2017, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (10 points)**.** Show, that the Banach space $\mathcal{C}[0,1]$ is separable. Deduce (without using an ONB), that the Hilbert space $L^2[0,1]$ is separable.
Remark: This especially shows that $L^2[0,1]$ is isomorphic to $\ell^2(\mathbb{N})$.

Exercise 2 (10 points)**.** Prove Bessel's inequality for arbitrary index sets I: For any orthonormal system $(e_i)_{i\in I}$ in a Hilbert space H it holds

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \qquad \forall x \in H$$

with equality if and only if $x = \sum_{i\in I}\langle x, e_i\rangle e_i$.
Note: We use Bessel's inequality to show Parseval's theorem. So this can't be used here.

Exercise 3 (10 points)**.** We consider $\ell^2 = \ell^2(\mathbb{N})$ together with the canonical ONB $(e_n)_{n\in\mathbb{N}}$. For $A \in \mathcal{L}(\ell^2)$ we define the *matrix coefficients* by $a_{ij} := \langle Ae_j, e_i\rangle$ for all $i,j \in \mathbb{N}$.

(a) Show that A is uniquely defined by its matrix coefficients.

(b) Show that for $C := \|A\|$ we have:

$$\sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |a_{ij}|^2 \leq C^2, \qquad \sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} |a_{ij}|^2 \leq C^2$$

(c) So to each operator we can associate an infinite matrix. Show, that the converse is not true. For this, construct a matrix $(b_{ij})_{i,j\in\mathbb{N}}$, such that we have inequality (b) for some $C \geq 0$, but the (b_{ij}) do *not* describe the matrix coefficients of a bounded, linear operator on ℓ^2.

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Exercise 4 (10 points)**.** Let H be a Hilbert space and $K \subseteq H$ a closed subspace. For $x + y \in K \oplus K^\perp$ we put $P(x + y) := x$.

(a) Show that $P \in \mathcal{L}(H)$ and compute $\|P\|$.

(b) Show that P is an orthogonal projection in the allgebraic sense, i.e. $P = P^* = P^2$.

(c) Show that $\mathrm{im}(P) = K$ and $\ker(P) = K^\perp$ (P *projects* onto K).

(d) Let conversely $P \in \mathcal{L}(H)$ be a projection in the sense of (b). Show that there is a closed subspace $K \subseteq H$, such that $P(x + y) = x$ for $x + y \in K \oplus K^\perp$.