UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
Prof. Dr. Moritz Weber
Stefan Jung

Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 8
submission: Tuesday, December 19 2017, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (10 points)**.** Let $A \in \mathcal{L}(H)$ be selfadjoint (i.e. $A^* = A$). Show:

$$\|A\| = \sup\{|\langle Ax, x\rangle| \mid \|x\| = 1\}$$

Hint for $\leq$*:* If $\|Ax\| \geq \|A\| - \varepsilon$ for $\|x\| = 1$, consider a suitable ONB $\{x, y, \dots\}$ of H and investigate the corresponding "coefficient matrix" of A.

Exercise 2 (10 points)**.** Let H, K be Hilbert spaces and $V : H \to K$ linear and bounded.

(a) Show, that the following definitions of "V is isometric" are equivalent (where $1 \in \mathcal{L}(H)$ is defined by $1x := x$):

(i) $V^*V = 1$, (ii) $\langle Vx, Vy\rangle = \langle x, y\rangle \ \forall x, y \in H$, (iii) $\|Vx\| = \|x\| \ \forall x \in H$

(b) Show that V is unitary (i.e. $V^*V = VV^* = 1$) if and only if V is a surjective isometry. (These are exactly the isomorphisms of Hilbert spaces, see 5.35 of the lecture.)

(c) A self-adjoint isometry is called *symmetry*. Show that V is a symmetry if and only if there are closed subspaces $H_+, H_- \subseteq H$ such that $H = H_+ \oplus H_-$ and $V(x_+ + x_-) = x_+ - x_-$ holds for all $X_+ \in H_+$ and $x_- \in H_-$.
Hint: Investigate $\frac{1}{2}(1 \pm V)$.

please turn the page

Exercise 3 (10 points)**.** On the last exercise sheet we associated to $A \in \mathcal{L}(\ell^2)$ a matrix of infinite size. An operator $A \in \mathcal{L}(\ell^2)$ is called *Hilbert-Schmidt operator*, if:

$$\|A\|_{\mathrm{HS}} := \left(\sum_{i,j\in\mathbb{N}} |a_{ij}|^2 \right)^{\frac{1}{2}} < \infty \tag{1}$$

The norm $\|\cdot\|_{\mathrm{HS}}$ is called *Hilbert-Schmidt norm.* Let $\|\cdot\|_\infty$ denote the operator norm.

(a) Show: If a matrix $(a_{ij})_{i,j\in\mathbb{N}}$ fulfills condition (1), there is an operator $A \in \mathcal{L}(\ell^2)$ with these matrix coefficients. In this case it holds $\|A\|_\infty \leq \|A\|_{\mathrm{HS}}$.

(b) Show that every Hilbert-Schmidt operator is compact, but not every compact operator is a Hilbert-Schmidt operator.

(c) Let A be a Hilbert-Schmidt operator and $B, C \in \mathcal{L}(\ell^2)$. Show that BAC is a Hilbert-Schmidt oerator with $\|BAC\|_{\mathrm{HS}} \leq \|B\|_\infty \|A\|_{\mathrm{HS}} \|C\|_\infty$. So the Hilbert-Schmidt operators form a two-sided ideal in $\mathcal{L}(H)$.

Exercise 4 (10 points)**.** Let H be a Hilbert space, $K \subseteq H$ a closed subspace and P the corresponding projection (sheet 7). Let $A \in \mathcal{L}(H)$. We say that K is *invariant for* A, if $AK \subseteq K$. Show that K is invariant for A and A^* if and only if $PA = AP$.