UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 9
submission: Tuesday, January 2 2018, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (20* points)**.** The *one-sided* shift $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by $Se_n := e_{n+1}$, where $(e_n)_{n\in\mathbb{N}}$ is the standard ONB of $\ell^2(\mathbb{N})$.

(a) Compute S^* and show that S is an isometry, but not a unitary. (This is possible only in the infinite dimensional case.)

(b) Show that $S - \lambda$ is invertible for $|\lambda| > 1$.

(c) Show that S has no eigenvalues. So there is no $\lambda \in \mathbb{C}$ and $x \in \ell^2(\mathbb{N})$ with $x \neq 0$ and $Sx = \lambda x$.

(d) Show that λ is an eigenvalue for S^* whenever $|\lambda| < 1$.

(e) Show that $\mathrm{Sp}(S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.

Exercise 2 (20* points)**.** Let (X, d) be a metric space. We say that in (X, d) the *balls are round* if for $x_1, x_2 \in X$ and $r_1, r_2 > 0$ with $d(x_1, x_2) = r_1 + r_2$ there is always exactly one $x_0 \in X$ lying in the intersection of the closed balls $\overline{B(x_1, r_1)}$ and $\overline{B(x_2, r_2)}$. So round balls of this radii intersect in exactly one point.

Santa Claus wants to put up a christmas tree but cannot deside where to buy the glitter balls in order to decorate the tree. He is balancing between *Banach's Beautiful Ball Deliveries* and *Hilbert's Ho-Ho-Homeshop*. Santa wants to assure that his grandchilden do not hurt themselves at the tree decoration, so he prefers round balls.

(a) Can Santa order the glitter balls wherever he wants (proof or counterexample)?

(b) Are Santa's grandchildren safe (at least from this perspective) if he chooses *Hilbert's Ho-Ho-Homeshop* (proof or counterexample)?

(c) Draw a christmas tree with typical glitter balls from *Banach's Beautiful Ball Deliveries* and another tree with typical glitter balls from *Hilbert's Ho-Ho-Homeshop*, both with lots of nice christmas decorations.

Hint: Use Theorem 5.10 from the lecture for (b) and coloured pencils for (c).

We wish all students a merry christmas, nice holidays and a happy new year.