UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 10
submission: Tuesday, January 9 2018, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (10 points)**.** let H be a Hilbert space. Show that for an operator $V \in \mathcal{L}(H)$ the following are equivalent:

(i) V is a *partial isometry*, i.e. $VV^*V = V$.

(ii) V^*V is an orthogonal projection (see Def. 7.9 and sheet 7).

(iii) VV^* is an orthogonal projection.

(iv) There is a closed subspace $K \subseteq H$, s.t. $V|_K : K \to H$ is isometric (i.e. $\|Vx\| = \|x\|$ for all $x \in K$) and $V|_{K^\perp} = 0$.

Exercise 2 (**20** points)**.** We consider

$$\ell^1(\mathbb{Z}) := \{(\alpha_n)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |\alpha_n| < \infty\}, \qquad \|(\alpha_n)_{n\in\mathbb{Z}}\|_1 := \sum_{n\in\mathbb{Z}} |\alpha_n|$$

with the *convolution*:

$$(\alpha_n)_{n\in\mathbb{Z}} * (\beta_n)_{n\in\mathbb{Z}} := (\gamma_n)_{n\in\mathbb{Z}}, \qquad \text{where } \gamma_n := \sum_{k\in\mathbb{Z}} \alpha_k \beta_{n-k}$$

(a) Show that $\ell^1(\mathbb{Z})$ with this multiplication is a commutative Banach algebra.

(b) Show that every character of $\ell^1(\mathbb{Z})$ is of the form $\varphi_z : A \to \mathbb{C}$, with $\varphi_z((\alpha_n)_{n\in\mathbb{N}}) := \sum_{n\in\mathbb{Z}} \alpha_n z^n$ for a suitale $z \in \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.
So we identify $\mathrm{Spec}(A) \cong \mathbb{T}$ as sets (we can even identify them as topological spaces).

Exercise 3 (10 points)**.** A C^*-algebra is called *simple*, if it contains no proper ideals (if $I \subseteq A$ is a closed, two-sided ideal, then it holds $I = 0$ or $I = A$). Show that $M_n(\mathbb{C})$ is simple. Deduce that $\mathrm{Spec}(M_n(\mathbb{C}))$ is empty.

While the set of complex homomorphisms for commutative Banach algebras carries all informations about the algebra, this is in general not true for non-commutative ones.