UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 11
submission: Tuesday, December 16 2017, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (15 points)**.** Let A be a unital C^*-algebra. Show:

(a) If $x \in A$ is invertible, then Sp $x^{-1} = \{\lambda^{-1} \mid \lambda \in \text{Sp } x\}$.

(b) If $u \in A$ is unitary (i.e. $u^*u = uu^* = 1$), then Sp $u \subseteq \mathbb{T} \subseteq \mathbb{C}$, where $\mathbb{T}$ as on sheet 10 is the unit circle in $\mathbb{C}$.

Show herefore that $\|u\| = \|u^*\| = 1$ and use (a).

(c) If $x \in A$ is selfadjoint, then Sp $x \subseteq \mathbb{R}$.

Show herefore that every spectral value λ from Sp x induces a spectral value of the unitary lift $u := e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$. Herefore, the element $z := \sum_{n=1}^{\infty} \frac{i^n(\lambda - x)^{n-1}}{n!}$ is helpful. Finally, use (b).

Exercise 2 (10 points)**.** let $A \in \mathcal{B}(H)$. Show that A is positive if and only if $\langle Ax, x\rangle \geq 0$ for all $x \in H$.
Hint: Show that $A - \lambda$ is bounded from below and surjective for $\lambda \notin [0, \infty)$.

Exercise 3 (15 points)**.** Let H be a Hilbert space and $A \in \mathcal{B}(H)$. We define $|A| := \sqrt{A^*A}$. (Why is this well-defined?)

(a) Show that $\ker|A| = \ker(A)$ and that the map $\Psi : \text{im}|A| \to \text{im}(A),\ |A|\xi \mapsto A\xi$ is well-defined and isometric. So it has an isometric extention $\Psi_0 : \overline{\text{im}|A|} \to \overline{\text{im}(A)}$.

(b) We define:

$$V = \begin{cases} \Psi_0, & \text{on } \overline{\text{im}|A|} \\ 0, & \text{on } \overline{\text{im}|A|}^{\perp} = \ker(A) \end{cases}$$

Show that V is a *partial isometry*, i.e. it holds $V = VV^*V$. Show further that V^*V is the projection onto $(\ker(A))^{\perp}$ and VV^* the projection onto $\overline{\text{im}(A)}$.

(c) Show that we can write A as $A = V|A|$ and that the partial isometry V is uniquely defined by $A = V|A|$ and $\ker(V) = \ker(A)$. This decomposition of A is called *polar decomposition* (analogously to the polar decomposition in $\mathbb{C}$).

(d) Show that V is unitary (so $V^*V = VV^* = 1$), if A is invertible.