UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6 – MATHEMATIK
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Exercises to the lecture 'Functional Analysis'
Winter term 2017/2018

sheet 12
submission: Tuesday, January 23, 2018, 2 pm
postbox of Vincent Preiß (basement of E2.5)

Exercise 1 (20 points). Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$ and $M_\alpha : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by $(x_n) \mapsto (\alpha_n x_n)$.

(a) Show that M_α is normal, that $Sp(\alpha) = \overline{\{\alpha_n \mid n \in \mathbb{N}\}}$ and that $f\big(M_\alpha\big) = M_{f(\alpha)}$ for every continuous function f on $Sp(M_\alpha)$. Here $f(\alpha) := \big(f(\alpha_n)\big)_{n\in\mathbb{N}}$.

(b) Characterize those α, for which M_α is compact.

Exercise 2 (10 points). Finish the proof of Proposition 10.27 of the lecture:

(a) Show that if $x \in A$ is positive, there is a positive element $y \in A$, such that $y^2 = x$ using functional calculus.

(b) Show that if $y_1, y_2 \in C^*(x, 1) \subseteq A$ are positive and such that $y_1^2 = y_2^2 = x$, they may be written as $f_1(x) = y_1$ and $f_2(x) = y_2$ for some continuous functions f_1 and f_2. Prove $f_1 = f_2$.

(c) Let $y \in A$ be positive such that $y^2 = x$. Use $C^*(x, 1) \subseteq C^*(y, 1) \cong C(Sp(y))$ and Stone-Weierstraß to show that $y \in C^*(x, 1)$.

(d) Conclude that every positive element in a unital C^*-algebra A has a unique positive square root. In particular, every positive operator in $B(H)$ has a unique positive square root.

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Let H be a Hilbert space and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of operators in $B(H)$. Let $T \in B(H)$. We define the following three *operator topologies*.

(i) We say that $(T_n)_{n\in\mathbb{N}}$ *converges in norm* to T, if $||T_n - T|| \to 0$ for $n \to \infty$.

(ii) We say that $(T_n)_{n\in\mathbb{N}}$ *converges strongly* to T, if $||T_n\xi - T\xi|| \to 0$ for $n \to \infty$ and all $\xi \in H$.

(iii) We say that $(T_n)_{n\in\mathbb{N}}$ *converges weakly* to T, if $\langle (T_n - T)\xi, \eta\rangle \to 0$ for $n \to \infty$ and all $\xi, \eta \in H$.

Exercise 3 (10 points)**.** We compare the three operator topologies.

(a) Prove that convergence in norm implies strong convergence, and that strong convergence implies weak convergence.

(b) For which of the three operator topologies, the norm is continuous? (Investigate the sequence of projections $T_n := P_n$ projecting onto n-th ONB vector.)

(c) For which of the three operator topologies, the involution is continuous? (Investigate $T_n := (S^*)^n$, where S is the unilateral shift.)

(d) Show for the three topologies: If $U \in B(H)$ and if $T_n \to T$, then $UT_n \to UT$ and $T_nU \to TU$, for $n \to \infty$.

Special exercise (100 Euro*)**.** Prove or disprove: There is a number $n \in \mathbb{N}$, a Hilbert space H and selfadjoint operators $u_{ij} \in B(H)$, $i, j \in \{1, \dots, n\}$ such that

(i) $$\sum_{k_1,k_2=1}^{n} u_{i_1k_1}u_{i_2k_2}u_{j_1k_1}u_{j_2k_2} = \sum_{k_1,k_2=1}^{n} u_{k_1i_1}u_{k_2i_2}u_{k_1j_1}u_{k_2j_2} = \delta_{i_1j_1}\delta_{i_2j_2} \text{ for all } i_1, i_2, j_1, j_2$$

(ii) and there are i, j, k, l such that $u_{ij}u_{kl} \neq u_{kl}u_{ij}$.

**to be split in case there are several correct solutions*