Sheet 3

Ex 1

a) d is a metric:

- $d(x,y) \geq 0$ clear

- $$d(x,y) = 0 \iff \underbrace{\sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}}_{\geq 0} = 0 \iff p_n(x-y) = 0 \ \forall n \in \mathbb{N}_0$$
$$\Rightarrow x - y = 0 \iff x = y$$

- $x = y \Rightarrow d(x,y) = 0$ clear

- Δ-inequality: Recall that $x \mapsto \frac{x}{1+x}$ is increasing on $[0,\infty)$

and for $(x_\lambda)_{\lambda \in \Lambda}$ with $x_\lambda \geq 0$: $\frac{x_\lambda}{1+x_\lambda} \xrightarrow{\lambda} 0 \iff x_\lambda \xrightarrow{\lambda} 0$

Now $\forall x,y,z \in X$:

$$d(x,z) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x-z)}{1+p_n(x-z)} \leq \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x-y) + p_n(y-z)}{1+p_n(x-y)+p_n(y-z)}$$

$$= \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)+p_n(y-z)} + \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(y-z)}{1+p_n(x-y)+p_n(y-z)}$$

$$\leq \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)} + \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(y-z)}{1+p_n(y-z)}$$

$$= d(x,y) + d(y,z)$$

$\square$

Ex 1

b) d induces same topology as semi-norm system

„$\Rightarrow$": Assume $x_\lambda \xrightarrow{\lambda} x$ w.r.t. d

i.e. $d(x_\lambda, x) \xrightarrow{\lambda} 0 \Leftrightarrow \sum_{n=0}^{\infty} \underbrace{2^{-n} \frac{p_n(x_\lambda - x)}{1+p_n(x_\lambda - x)}}_{\geq 0} \xrightarrow{\lambda} 0$

$$\Rightarrow \frac{p_n(x_\lambda - x)}{1+p_n(x_\lambda - x)} \xrightarrow{\lambda} 0 \quad \forall n \in \mathbb{N}_0$$

$$\Leftrightarrow p_n(x_\lambda - x) \xrightarrow{\lambda} 0 \quad \forall n \in \mathbb{N}_0$$

$$\Leftrightarrow x_\lambda \xrightarrow{\lambda} x \quad \text{w.r.t. } \{p_0, p_1, \dots\}$$

„$\Leftarrow$": Assume $x_\lambda \xrightarrow{\lambda} x$ w.r.t. $\{p_0, p_1, \dots\}$

Let $\varepsilon > 0$ be given. $\Rightarrow \exists N \geq 0 : \sum_{n \geq N} 2^{-n} < \frac{\varepsilon}{2}$

$\forall n$ with $0 \leq n \leq N$: $\exists \hat{\lambda}_n : \forall \lambda \geq \hat{\lambda}_n \; 2^{-n} \frac{p_n(x_\lambda - x)}{1+p_n(x_\lambda - x)} < \frac{\varepsilon}{2(N+1)}$

Let $\hat{\lambda}$ s.t. $\hat{\lambda} \geq \lambda_i \; \forall 0 \leq i \leq N$. Then $\forall \lambda \geq \hat{\lambda}$:

$$d(x_\lambda, x) \leq \sum_{n=0}^{N} \underbrace{2^{-n} \frac{p_n(x_\lambda - x)}{1+p_n(x_\lambda - x)}}_{< \frac{\varepsilon}{2(N+1)}} + \underbrace{\sum_{n=N}^{\infty} 2^{-n} \underbrace{\frac{p_n(x_\lambda - x)}{1+p_n(x_\lambda - x)}}_{\leq 1}}_{< \frac{\varepsilon}{2}}$$

$$< (N+1) \frac{\varepsilon}{2(N+1)} + \frac{\varepsilon}{2} = \varepsilon \quad \boxtimes$$

Ex 11

c) $X = C^\infty[0,1]$, $p_n(f) = \|f^{(n)}\|_\infty$

t.s.: X complete w.r.t. topology induced by $\{p_0, p_1, \ldots\}$

proof: For any $n \in \mathbb{N}_0$

$(f_\lambda)_{\lambda\in\Lambda}$ Cauchy w.r.t. topology above $\Rightarrow (f_\lambda)_{\lambda\in\Lambda}$ Cauchy w.r.t. p_n

$\Rightarrow (f_\lambda^{(n)})_{\lambda\in\Lambda}$ Cauchy in $(C[0,1], \|\cdot\|_\infty)$

$\Rightarrow \quad f_\lambda^{(n)} \xrightarrow{\lambda} g_n$ uniformly in $(C[0,1], \|\cdot\|_\infty)$

$\overset{\text{Hint}}{\Rightarrow} \quad g_n' = g_{n+1}$ (especially g_n' exists)

$\overset{\text{induction}}{\Rightarrow} \quad g_n = g_0^{(n)}$

Define $f := g_0$. Then $f \in C^\infty[0,1]$ and $(f_\lambda)_{\lambda\in\Lambda}$

converges to f w.r.t. topology induced by $\{p_0, p_1, \ldots\}$

as $\forall n \in \mathbb{N}$: $p_n(f_\lambda - f) \xrightarrow{\lambda} 0$.

Ex21

① $\mathbb{Q}$ countable $\Rightarrow$ $\mathbb{Q}^n$ countable $\Rightarrow$ $\bigcup_{n\in\mathbb{N}} \mathbb{Q}^n$ countable

$\Rightarrow \{(a_n)\in f \mid a_n \in \mathbb{Q}\ \forall n\}$ countable

as $\{(a_n)\in f \mid a_n\in\mathbb{Q}\ \forall n\}$ dense in f

and f dense in $(c_0, \|\cdot\|_\infty)$:

$\{(a_n)\in f \mid a_n\in\mathbb{Q}\ \forall n\}$ dense in $(c_0, \|\cdot\|_\infty)$

So $(c_0, \|\cdot\|_\infty)$ separable

② Assume $(\ell^\infty, \|\cdot\|_\infty)$ separable.

Let $\left(\left(a_n^{(k)}\right)_{n\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ dense subset of $\ell^\infty, \|\cdot\|_\infty$

Now define sequence (b_n) by:

$$b_n = \begin{cases} 0, & |a_n^{(n)}| \geq 1 \\ 1, & a_n^{(n)} = 0,\ a_n^{(n)} \in (-1, 0) \\ -1, & a_n^{(n)} \in (0,1) \end{cases}$$

Then $(b_n)\in(\ell^\infty, \|\cdot\|_\infty)$ but $\|(b_n) - (a_n^{(k)})\|_\infty \geq 1\ \forall k\in\mathbb{N}$

So $\left((a_n^{(k)})_{n\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ not dense in $(\ell^\infty, \|\cdot\|_\infty)$ ↯

Ex 3|

Let $(e_i)_{i \in I}$ be Hamel-basis of infinite-dimensional normed space V. (so $|I| = \infty$)

WLOG $\mathbb{N} \subseteq I$

Define now $T(e_i) = \begin{cases} n \cdot e_n , & i = n \\ 0 , & i \notin \mathbb{N} \end{cases}$

Then T on V well-defined by linear extension

and $\|T(e_n)\| = \|n \cdot e_n\| = n \cdot \|e_n\| \quad \forall n \in \mathbb{N}$

so T unbounded.

Ex 4| Let $x \in E$ be given.

Then $\forall f \in E'$ with $\|E'\| \leq 1$ we have $|f(x)| \leq \|f\| \cdot \|x\| \leq \|x\|$.

On the other side:

Cor. 2.6.

$\Rightarrow$ Define $f : \mathbb{C}x \to \mathbb{C} : \alpha x \mapsto \alpha \|x\|$

(so $\|f\| = 1$ and linear, bounded.

Then $\exists\ \tilde{f} \in E'$ extension of f, $\|\tilde{f}\| = \|f\| = 1$ and $\tilde{f}(x) = \|x\|$.

Especially we have $\sup\{|\tilde{f}(x)| \mid \tilde{f} \in E', \|\tilde{f}\| \leq 1\} \geq \|x\|$

so all together "$=$".

Ex 51

Define $\tilde{L}$ on subspace of constant sequences by

$$\tilde{L}(c) \mapsto c$$

Then $\tilde{L}(c) \leq \limsup_{n\to\infty} c = c$ and $\tilde{L}$ bounded

If $q((a_n)) = \limsup_{n\to\infty} a_n$ is sublinear functional then

thm 2.3

$\Rightarrow$ $\exists$ extension L of $\tilde{L}$, L linear with $L((a_n)) \leq \limsup_{n\to\infty} a_n \leq \sup_{n\in\mathbb{N}} |a_n|$

Especially L is bounded.

Furthermore:

$$L(-(a_n)) \leq \limsup_{n\to\infty} (-a_n) = -\liminf_{n\to\infty} a_n$$

$$\Leftrightarrow \liminf_{n\to\infty} a_n \leq L((a_n)).$$

Remains to show: $\limsup_{n\to\infty}$ sublinear functional

For $\lambda \geq 0$ we have: $\limsup_{n\to\infty} (\lambda(a_n)) = \inf_{n\geq 0} (\sup_{k\geq n} \lambda a_k)$

$$= \inf_{n\geq 0} (\lambda \cdot \sup_{k\geq n} a_k) = \lambda \inf_{n\geq 0} (\sup_{k\geq n} a_k)$$

$$= \lambda \limsup_{n\to\infty} a_n.$$

$\limsup_{n\to\infty} ((a_n)+(b_n)) = \inf_{n\geq 0} (\sup_{k\geq n} a_k + b_k) \leq \inf_{n\geq 0} (\sup_{k\geq n} a_k + \sup_{k\geq n} b_k) = \inf_{n\geq 0} \sup_{k\geq n} a_k + \inf_{n\geq 0} \sup_{k\geq n} b_k = q((a_n)) + q((b_n))$

$\varnothing$

Especially we have

① $L(1) = \tilde{L}(1) = 1 \rightsquigarrow \quad \|L\| \geq 1$

② If $a_n \geq 0 \ \forall n \in \mathbb{N}$ $\quad \liminf\limits_{n\to\infty} a_n \geq 0$ so $\quad L((a_n)) \geq 0$

③ $L((a_n)) \leq \sup\limits_{n\in\mathbb{N}} |a_n| = \|(a_n)\|_\infty \rightsquigarrow \quad \|L\| \leq 1$

①+③ $\Rightarrow \quad \|L\| = 1$

④ For all convergent (a_n):

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n \leq L((a_n)) \leq \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n .$$

<u>Ex 6)</u>: Consider $\quad a_n^{(k)} := \begin{cases} (-1)^k, & n=1 \\ \frac{1}{k\cdot n}, & n\geq 2 \end{cases}$

Then $K := \{(a_n^{(k)})_{n\in\mathbb{N}} \mid k\in\mathbb{N}\} \cup \{(\pm\delta_{n,1})_{n\in\mathbb{N}}\}$

is compact but

$$(d_n^{(k)}) := \frac{1}{2}(a_n^{(2k)}) + \frac{1}{2}(a_n^{(2k-1)}) \text{ fulfills}$$

$$d_n^{(k)} = \begin{cases} 0, & n=1 \\ \frac{1}{kn}, & n\geq 2 \end{cases}$$

so $\quad (d_n^{(k)}) \xrightarrow{k\to\infty} (0) \notin K$ as $b_1 = \pm 1 \ \forall (b_n) \in K$.

☺