Sheet 8

Ex 11

Let WLOG $\|A\|=1$ and let $0<\varepsilon<1$ be given.

Then $\exists x\in H$, $\|x\|=1$ with $\|Ax\|>1-\varepsilon$

If $Ax \parallel x$ then $\sup\limits_{\|x\|=1} |\langle Ax,x\rangle| > 1-\varepsilon$

If $Ax \nparallel x$ define $$y := \frac{Ax - \langle Ax,x\rangle x}{\|Ax - \langle Ax,x\rangle x\|}$$

Consider now $\tilde{A} := \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ with $\alpha := \langle A\cancel{x},\cancel{x}\rangle \in \mathbb{R}$

$\gamma := \langle Ay, y\rangle \in \mathbb{R}$

$\beta = \underbrace{\langle Ax, y\rangle}_{>0} = \langle Ay, x\rangle$

quasi: $A = \begin{pmatrix} \alpha & \beta & 0 & \cdots \\ \beta & \gamma & \cdots & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}$

$\leadsto \|\tilde{A}\| \geq \|Ax\|$ as $\|\tilde{A}\binom{1}{0}\| = \sqrt{\alpha^2+\beta^2} = \|Ax\|$

As $\tilde{A}$ diagonalizable: $\exists v = \binom{v_1}{v_2}$, $\|v\|=1$, such that $\tilde{A}v = \lambda v$ for $|\lambda| = \|\tilde{A}\| \geq \|Ax\|$

$\Rightarrow |\langle A(v_1x+v_2y), (v_1x+v_2y)\rangle| \geq \|\tilde{A}v\| = |\lambda| \geq \|Ax\| > 1-\varepsilon$

So $\sup\limits_{\|x\|=1} |\langle Ax,x\rangle| \geq 1$

As „$\leq$" is clear by C.S. and $\|Ax\| \leq \|A\|\cdot\|x\|$ we have equality.

⌈Note: Proof works also in non-separable case. The matrices $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ and $\begin{pmatrix} \alpha & \beta & \\ \beta & \gamma & \\ & & \ddots \end{pmatrix}$ just help to illustrate but in principle you can define v_1, v_2 only knowing α, β, γ and then compute $|\langle A(v_1x+v_2y), v_1x+v_2y\rangle|$ ⌋

Ex21

a)

(i) $\Rightarrow$ (ii): $\quad \langle Vx, Vy\rangle = \langle \underbrace{V^*V}_{=\mathbb{1}} x, y\rangle = \langle x, y\rangle$

(ii) $\Rightarrow$ (iii) $\quad \|Vx\|^2 = \langle Vx, Vx\rangle = \langle x, x\rangle = \|x\|^2$

(iii) $\Rightarrow$ (i): $\quad$ Assume $\|Vx\| = \|x\|$ $\forall x \in H$.

$$\|V^*x\| = \sup_{\|y\|=1} |\langle V^*x, y\rangle| = \sup_{\|y\|=1} |\langle x, Vy\rangle|$$

$$\underset{C.S.}{\leq} \|x\| \cdot 1 = \|x\| \qquad \forall x \in H$$

$\Rightarrow \|V^*Vx\| \leq 1$

Now: $\langle V^*Vx, x\rangle = \langle Vx, Vx\rangle = \|Vx\|^2 = \|x\|^2$

so $V^*Vx = x + y$ with $y \in \{x\}^\perp$

but $\|x+y\|^2 \underset{\text{Pythagoras}}{=} \|x\|^2 + \|y\|^2 \leq \|x\|^2$ so $y = 0$.

$\rightsquigarrow$ $V^*Vx = x$ $\forall x$, i.e. $V^*V = \mathbb{1}$

b) If V unitary then V surjective and isometric ✓

If V isometry then by (ii) in a) V injective.
Together with surjectivity we have that V is bijective so by the open mapping theorem V is invertible.

As $V^*V = \mathbb{1}$ by (i) in a) we must have $V^* = V^{-1}$

(as $V^*V = \mathbb{1} \Rightarrow V^*VV^{-1} = V^{-1} \Leftrightarrow V^* = V^{-1}$).

c) Consider $\frac{1}{2}(\mathbb{1} \pm V) = P_\pm$

$$P_\pm = P_\pm^* \quad \text{and} \quad P_\pm^2 = \frac{1}{4}(1 \pm 2V + V^2) = \frac{1}{2}(1 \pm V) = P_\pm$$

so $\exists$ closed linear subspaces $H_\pm$ s.t. $P_\pm$ projects onto $H_\pm$.

We have $P_+ + P_- = \mathbb{1}$ so $H_+ + H_- = H$

and $\forall x_\pm \in H_\pm$: $P_\mp(x_\pm) = P_\mp P_\pm(x_\pm) = 0(x_\pm) = 0$

so $H_+ \perp H_-$

$\rightsquigarrow$ $H = H_+ \oplus H_-$

At last: $\forall x_\pm \in H_\pm$: $\cancel{P_\pm(x_\pm)}$

$$\frac{1}{2}(\mathbb{1} \pm V)(x_\pm) = x_\pm$$

$$\Rightarrow \quad V(x_\pm) = \pm x_\pm$$

$$\Rightarrow \quad V(x_+ + x_-) = x_+ - x_-$$

Ex31

a) Let $v = \sum_{k=1}^{\infty} \beta_k e_k \in \ell^2$

Try to define $A\left(\sum_{k=1}^{\infty} \beta_k e_k\right) := \sum_{i=1}^{\infty} \underbrace{\left(\sum_{j=1}^{\infty} \alpha_{ij} \beta_j\right)}_{|\cdot| \underset{C.S.}{\le} \sqrt{\sum_{j=1}^{\infty} |\alpha_{ij}|^2} \cdot \sqrt{\sum |\beta_j|^2}} e_i$

So $$\|Av\|^2 = \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \alpha_{ij} \beta_j \right|^2$$

$$\le \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_{ij}|^2 \cdot \|v\|^2 = \|A\|_{HS}^2 \, \|v\|^2$$

As A is clearly linear, we have that A is well-defined and the computation above shows $\|A\|_{\infty} \le \|A\|_{HS}$.

b) Let A be HS-operator. Consider any sequence

$$\left(v^{(n)} := \sum_{k=1}^{\infty} \beta_k^{(n)} e_k\right)_{n \in \mathbb{N}} \quad \text{in} \quad B(0,1) \subseteq H.$$

By a diagonal sequence argument we can assume that $\left(\beta_k^{(n)}\right)_{n \in \mathbb{N}}$ converges $\forall k \in \mathbb{N}$.

We show that $A(v^{(n)})$ is Cauchy. (Assume $\|A\|_{\infty} > 0$ as „$=0$" trivial)

Let $\varepsilon > 0$ be given.

As $\sum_{i,j=1}^{\infty} |\alpha_{ij}|^2 < \infty$ we have $\sum_{i=N}^{\infty} \sum_{j=1}^{\infty} |\alpha_{ij}|^2 \xrightarrow[\infty]{N} 0$

So let $K \in \mathbb{N}$ s.t. $\sum_{i=K+1}^{\infty} \sum_{j=1}^{\infty} |\alpha_{ij}|^2 < \frac{\varepsilon^2}{16}$.

As $\sum_{k=1}^{K} \beta_k^{(n)} e_k$ converges $\exists N \in \mathbb{N}: \forall m,n \geq N: \|\sum_{k=1}^{K} (\beta_k^{(n)} - \beta_k^{(m)}) e_k\| < \frac{\varepsilon}{2\|A\|_\infty}$

But then we have $\forall m,n \geq N$:

$$\| A v^{(n)} - A v^{(m)} \| = \| A (v^{(n)} - v^{(m)}) \|$$

$$\leq \| A (\sum_{k=1}^{K} (\beta_k^{(n)} - \beta_k^{(m)}) e_k) \| + \| A (\sum_{j=K+1}^{\infty} (\beta_j^{(n)} - \beta_j^{(m)}) e_j) \|$$

$$\leq \|A\|_\infty \cdot \frac{\varepsilon}{2\|A\|_\infty} + \| \sum_{i=1}^{\infty} \sum_{j=K+1}^{\infty} \alpha_{ij} (\beta_j^{(n)} - \beta_j^{(m)}) e_i \|$$

$$= \frac{\varepsilon}{2} + \sqrt{\sum_{i=1}^{\infty} | \sum_{j=K+1}^{\infty} \alpha_{ij} (\beta_j^{(n)} - \beta_j^{(m)}) |^2}$$

$$\underset{C.S}{\leq} \frac{\varepsilon}{2} + \sqrt{\underbrace{\sum_{i=1}^{\infty} \sum_{j=K+1}^{\infty} |\alpha_{ij}|^2}_{\leq \frac{\varepsilon^2}{16}} \cdot \underbrace{\sum_{j=K+1}^{\infty} |\beta_j^{(n)} - \beta_j^{(m)}|^2}_{\leq \|v^{(n)} - v^{(m)}\|^2 \leq 4}}$$

$$\leq \frac{\varepsilon}{2} + \sqrt{\frac{\varepsilon^2}{4}}$$

$$= \varepsilon$$

So A is compact.

Conversely, consider $A \in \mathcal{L}(\ell^2)$ defined by $e_n \mapsto \frac{1}{\sqrt{n}} e_n$

Obviously $\|A\| = 1$ and $\sum_{i,j=1}^{\infty} |\alpha_{ij}|^2 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

But A is compact: Consider sequence $(v^{(n)})_{n\in\mathbb{N}}$ as before and again assume componentwise convergence.

For $\varepsilon > 0$ let $K > \frac{16}{\varepsilon^2}$

and let $N \in \mathbb{N}$: $\forall m,n \geq N$: $\|\sum_{k=1}^{K} (\beta_k^{(n)} - \beta_k^{(m)}) e_k\| \leq \frac{\varepsilon}{2}$

Then we have $\forall m,n \geq N$:

$$\|Av^{(n)} - Av^{(m)}\| = \|A(v^{(n)} - v^{(m)})\|$$

$$\leq \underbrace{\|A \sum_{k=1}^{K} (\beta_k^{(n)} - \beta_k^{(m)}) e_k\|}_{\leq \frac{\varepsilon}{2}} + \underbrace{\|A(\sum_{k=K+1}^{\infty} (\beta_k^{(n)} - \beta_k^{(m)}) e_k)\|}_{= \|\sum_{k=K+1}^{\infty} \frac{1}{\sqrt{k}} (\beta_k^{(n)} - \beta_k^{(m)}) e_k\|}$$

$$\leq \frac{\varepsilon}{2} + \underbrace{\frac{1}{\sqrt{K}}}_{\leq \sqrt{\frac{\varepsilon^2}{16}} = \frac{\varepsilon}{4}} \underbrace{\|\sum (\beta_k^{(n)} - \beta_k^{(m)}) e_k\|}_{\leq 2}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So A compact.

c) Let A HS-operator on ℓ^2 and $B \in \mathcal{L}(\ell^2)$

We have $\|A\|_{HS}^2 = \sum_{i,j=1}^{\infty} |\langle Ae_i, e_j \rangle|^2$

$$= \sum_{i=1}^{\infty} \|Ae_i\|^2$$

$$\Rightarrow \quad \|BA\|_{HS}^2 = \sum_{i=1}^{\infty} \|BAe_i\|^2 \leq \|B\|_\infty^2 \sum_{i=1}^{\infty} \|Ae_i\|^2 = \|B\|_\infty \cdot \|A\|_{HS}$$

As $\|B^*\|_\infty = \|B\|_\infty$ and $\|A^*\|_{HS} = \|A\|_{HS}$

$$\|AB\|_{HS} = \|B^* A^*\|_{HS} \leq \|B^*\|_\infty \cdot \|A^*\|_{HS} = \|B\|_\infty \|A\|_{HS}$$

$$\Rightarrow \quad \|BAC\|_{HS} = \|B(AC)\|_{HS} \leq \|B\|_\infty \cdot \|AC\|_{HS}$$

$$\leq \|B\|_\infty \cdot \|A\|_{HS} \cdot \|C\|_\infty$$

(especially $\|BAC\|_{HS} < \infty$ so BAC is HS-operator).

Ex41

"⇒" Let K be invariant for A and A^*.

For $y \in K^\perp$, $x \in K$ we have

$$\langle Ay, x \rangle = \langle y, \underbrace{A^* x}_{\in K} \rangle = 0 \quad \text{so} \quad Ay \in K^\perp$$

so $K^\perp$ invariant for A.

Now: $\underbrace{PAx}_{\in K} = A\underbrace{x}_{=Px} = APx$

$$\underbrace{PAy}_{\in K^\perp} = 0 = AP\underbrace{y}_{=0}$$

so alltogether $AP(z) = PA(z)$ $\forall z \in H \Rightarrow AP = PA$

"⇐" Let $AP = PA$, then $A(1-P) = (1-P)A$

with $1-P$ projection onto $K^\perp$.

Now for $x \in K$, $y \in K^\perp$.

$$Ax = APx = P(Ax) \in K \quad (\checkmark)$$

$$Ay = A(1-P)(y) = (1-P)Ay \in K^\perp$$

$$\Rightarrow \langle A^* x, y \rangle = \langle x, \underbrace{Ay}_{\in K^\perp} \rangle = 0 \quad \text{so} \quad A^* x \in (K^\perp)^\perp = K \quad (\checkmark)$$