Sheet 10

Ex. 1

(i) ⇒ (ii): $(V^*V)^* = V^*V$ (✓)

$$(V^*V)^2 = V^*\underbrace{VV^*V}_{=V} = V^*V \qquad (\checkmark)$$

(ii) ⇒ (iii): $(VV^*)^* = VV^*$ (✓)

$$(VV^*)^3 = V\underbrace{V^*VV^*V}_{=V^*V}V^* = VV^*VV^* = (VV^*)^2$$

$$\Rightarrow ((VV^*)^2 - (VV^*))((VV^*)^2 - (VV^*))^*$$

$$= \underbrace{(VV^*)^4}_{\substack{=(VV^*)(VV^*)^3 \\ =(VV^*)^3 \\ =(VV^*)^2}} - 2\underbrace{(VV^*)^3}_{=(VV^*)^2} + (VV^*)^2$$

$$= 0$$

so $(VV^*)^2 - VV^* = 0$

i.e. $(VV^*)^2 = VV^*$ (✓)

(iii) ⇒ (ii): See (ii) ⇒ (iii) and switch roles of V and V^*.

(iii) + (ii) $\Rightarrow$ (iv): Define $K := V^*V(H)$

$\rightsquigarrow$ K closed linear subspace as V^*V projection

$\Rightarrow \forall x \in K: \|x\|^2 = \langle x, x\rangle = \langle V^*Vx, x\rangle = \|Vx\|^2$

i.e. $V|_K$ isometric

$\Rightarrow \forall x \in K^\perp: \; 0 = \langle V^*Vx, x\rangle = \langle Vx, Vx\rangle = \|Vx\|^2$

i.e. $V|_{K^\perp} = 0$

(iv) $\Rightarrow$ (iii) + (ii): $\forall x \in K^\perp: \; V^*Vx = 0$

$\forall x \in K: \; \langle V^*Vx, x\rangle = \langle Vx, Vx\rangle = \langle x, x\rangle$

$\Rightarrow V^*Vx = x + x^\perp$ with $x^\perp \in \{x\}^\perp$

As $\|V\| \leq 1$ also $\|V^*V\| \leq 1$ so $\|x^\perp\| = 0$, i.e. $V^*Vx = x$.

$\Rightarrow V^*V$ projection (onto K).

(ii) $\Rightarrow$ (i): Let $K := V^*V(H)$

$\Rightarrow \forall x \in K: \; V\underbrace{V^*Vx}_{= x} = Vx$

$\Rightarrow \forall x \in K^\perp: \; V^*Vx = 0 \Rightarrow \langle V^*Vx, x\rangle = 0$

$\Rightarrow \langle Vx, Vx\rangle = 0$

$\Rightarrow \|Vx\| = 0$

$\Rightarrow 0 = VV^*Vx = Vx$

So all together $VV^*V = V$.

Ex21

a) We already know: $\ell^1(\mathbb{Z})$ Banach space

$*$ is commutative, bilinear, associative

Now it holds: $\| (\alpha_n) * (\beta_n) \|_1 = \| (\gamma_n) \|_1$

$$= \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \alpha_k \beta_{n-k} \right|$$

$$\leq \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\alpha_k| |\beta_{n-k}|$$

$$= \sum_{n \in \mathbb{Z}} \| (\alpha_n) \|_1 \, |\beta_{n-k}|$$

$$= \| (\alpha_n) \|_1 \cdot \| (\beta_n) \|_1 .$$

b) Let φ be charact. of $\ell^1(\mathbb{Z})$

Define $e_i := (\delta_{i,n})_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$

Define $z := \varphi(e_1)$

$\varphi(e_0) = 1 = z^0$ as $e_0 = \mathbb{1}$

$\varphi(e_n) = z^n$ as $e_n * e_m = e_{n+m}$ $\forall n, m \in \mathbb{Z}$.

$|z| = 1$ as otherwise $|\varphi(e_1)| = |z| > 1$ or $|\varphi(e_{-1})| = |z^{-1}| > 1$.

$\Rightarrow \quad \forall\, (\alpha_n) \in \ell^1(\mathbb{Z})$:

$$\varphi(\alpha_n) = \varphi\Big(\lim_{\substack{N \subseteq \mathbb{Z} \\ N \text{ finite}}} \Big(\sum_{n \in N} \alpha_n e_n \Big)\Big)$$

$$= \lim_{\substack{N \subseteq \mathbb{Z} \\ N \text{ finite}}} \sum_{n \in N} \alpha_n z^n = \sum_{n \in \mathbb{Z}} \alpha_n z^n .$$

$\boxtimes$

Ex 31 t.s.: $\mathcal{M}_n(\mathbb{C})$ simple.

Let I ideal in $\mathcal{M}_n(\mathbb{C})$. $I \neq \{0\}$.

Let $M \in I$ and $m_{ij} \neq 0$ for suitable $1 \leq i,j \leq n$.

$$\Rightarrow \quad \frac{1}{m_{ij}} E_{ki} M E_{jl} = E_{kl} \in I$$

$\forall k,l$

$\Rightarrow$ span $\{ E_{kl} \mid 1 \leq k,l \leq n \} > \subseteq I$

$\Rightarrow I = \mathcal{M}_n(\mathbb{C})$.

If $n = 1$ then $\varphi : \mathcal{M}_1(\mathbb{C}) \to \mathbb{C}$; $(m) \Rightarrow m$ is a character.

If $n \geq 2$ then $\ker(\varphi)$ is ideal in $\mathcal{M}_n(\mathbb{C})$

$\varphi(\mathbb{1}) = 1$

$\Rightarrow \ker(\varphi) = \{0\}$

$\Rightarrow \dim(\mathcal{M}_n(\mathbb{C})) = \dim(\mathbb{C}) = 1$ ↯