

Assignments for the lecture Complex Analysis II Winter term 2017/2018

Assignment 1A

for the tutorial on *Tuesday, November 7, 4:15 pm* (in Seminar Room 10)

Problem 1. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and consider $f \in \mathcal{O}(\Omega)$. Prove that

$$\det \begin{pmatrix} \frac{\partial u}{\partial x}(z) & \frac{\partial u}{\partial y}(z) \\ \frac{\partial v}{\partial x}(z) & \frac{\partial v}{\partial y}(z) \end{pmatrix} = |f'(z)|^2 \quad \text{for all } z \in \Omega,$$

where $u := \operatorname{Re}(f) : \Omega \to \mathbb{R}$ and $v := \operatorname{Im}(f) : \Omega \to \mathbb{R}$, as usual.

Use this together with the real inverse function theorem in order to prove that f is injective in an open neighborhood of any given point $z_0 \in \Omega$ if and only if $f'(z_0) \neq 0$ holds.

Problem 2. Let us denote by \mathcal{T} the set of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ that satisfy f(0) = 0 and f'(0) = 1 and that have no zeros on $\mathbb{D} \setminus \{0\}$.

(i) Let $f \in \mathcal{T}$ be given. Prove that there exists a unique function $g \in \mathcal{O}(\mathbb{D})$, such that

$$g(z)^2 = f(z^2)$$
 for all $z \in \mathbb{D}$ and $g'(0) = 1$.

We call g the square root transform of f.

Hint: Establish first the existence of a function $\tilde{f} \in \mathcal{O}(\mathbb{D})$ without zeros and with the property that $f(z^2) = z^2 \tilde{f}(z)$ holds for all $z \in \mathbb{D}$. Finally, show that \tilde{f} admits a holomorphic square root, i.e., there is a function $\tilde{g} \in \mathcal{O}(\mathbb{D})$ that satisfies $\tilde{f}(z) = \tilde{g}(z)^2$ for all $z \in \mathbb{D}$. Use this to construct the desired square root transform g of f.

(ii) Prove that the square root transform g of any $f \in \mathcal{T}$ is an odd function (i.e., it satisfies g(-z) = -g(z) for all $z \in \mathbb{D}$) and belongs to \mathcal{T} .