UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Dr. Roland Friedrich
Dr. Tobias Mai

Assignments for the lecture *Complex Analysis II*
Winter term 2017/2018

Assignment 1B
for the tutorial on *Tuesday, November 7, 4:15 pm* (in Seminar Room 10)

Problem 1. Consider the set $\mathcal{T}$ that was defined in Problem 2, Assignment 1A. Moreover, in accordance with Definition 0.1 (ii) in the lecture notes, we denote by $\mathcal{S}$ the set of all schlicht holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ that satisfy $f(0) = 0$ and $f'(0) = 1$.

(i) Prove that $\mathcal{S}$ is contained in $\mathcal{T}$.

(ii) Show that the square root transform g of any function $f \in \mathcal{S}$ also belongs to $\mathcal{S}$.

(iii) Let $g \in \mathcal{T}$ be the square root transform of $f \in \mathcal{T}$ (recall from Problem 2 (ii), Assignment 1A that g is an odd function) and consider their power series expansions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = z + \sum_{n=1}^{\infty} \alpha_{2n+1} z^{2n+1}.$$

Prove that

$$a_n = \sum_{k=1}^{n} \alpha_{2(n-k)+1}\alpha_{2k-1} \qquad \text{for all } n \in \mathbb{N},$$

where we put in addition $a_1 := 1$ and $\alpha_1 := 1$.

Problem 2. For any fixed $\theta \in \mathbb{R}$, consider the *rotated Koebe function* k_θ, i.e. the function

$$k_\theta :\ \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{(1 - ze^{i\theta})^2}.$$

(i) Show that k_θ belongs to $\mathcal{S}$.

(ii) Compute the power series expansion of k_θ on $\mathbb{D}$.

(iii) Prove that $k_0(\mathbb{D}) = \mathbb{C}\backslash(-\infty, -\frac{1}{4}]$. How does $k_\theta(\mathbb{D})$ look like?

Hint: At first, show that $\psi_1(z) := \frac{1+z}{1-z}$ defines a biholomorphic map $\psi_1 : \mathbb{D} \to \mathbb{H}$ between the unit disc $\mathbb{D}$ and the right complex half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$. Secondly, study the mapping $\psi_2 : \mathbb{H} \to \mathbb{C}$ that is defined by $\psi_2(z) := \frac{1}{4}(z^2 - 1)$. Finally, compute their composition $\psi_2 \circ \psi_1$ and use the previous observations to determine $k_0(\mathbb{D})$. In order to find $k_\theta(\mathbb{D})$, check that $k_\theta(z) = e^{-i\theta} k_0(e^{i\theta} z)$ for $z \in \mathbb{D}$.