UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture *Complex Analysis II*
Winter term 2017/2018

Assignment 2A
for the tutorial on *Tuesday, November 21, 4:15 pm* (in Seminar Room 10)

Our goal is the proof of the following theorem, which strengthens the assertion of the Bieberbach conjecture for functions in $\mathcal{S}$ that have convex range.

Theorem. *Let $g \in \mathcal{S}$ be given and consider its power series expansion*

$$g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad \text{for } z \in \mathbb{D}.$$

If the image $g(\mathbb{D})$ is convex, then the coefficients of g satisfy $|b_n| \leq 1$ for all $n \in \mathbb{N}$.

Recall that an arbitrary set $C \subseteq \mathbb{C}$ is said to be *convex*, if $tz_1 + (1-t)z_2 \in C$ holds for any choice of points $z_1, z_2 \in C$ and for each $t \in [0,1]$.

Problem 1. Let $f, g : \mathbb{D} \to \mathbb{C}$ be two holomorphic functions that satisfy $f(0) = 0$ and $g(0) = 0$. We consider their power series expansions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad \text{for } z \in \mathbb{D}.$$

Prove the following assertions:

(i) If g is schlicht and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ holds, then $|a_1| \leq |b_1|$.

Hint: Apply the Schwarz lemma.

(ii) If we suppose in addition to the assumptions made in (i) that $g(\mathbb{D})$ is convex, then $|a_m| \leq |b_1|$ holds for all $m \in \mathbb{N}$.

Hint: For fixed $m \in \mathbb{N}$, put $\zeta := \exp(\frac{2\pi i}{m})$ and consider the function

$$h : \mathbb{D} \to \mathbb{C}, \; re^{i\theta} \mapsto \frac{1}{m} \sum_{k=1}^{m} f\big(\zeta^k r^{\frac{1}{m}} e^{i\frac{\theta}{m}}\big).$$

Show that h is well-defined and holomorphic and compute its power series expansion.

Problem 2.

(i) Prove the theorem with the help of the results that were obtained in Problem 1.

(ii) Are the inequalities given in the theorem sharp, i.e., is there a function $g \in \mathcal{S}$ with convex image $g(\mathbb{D})$, such that $|b_n| = 1$ holds for all $n \in \mathbb{N}$?