UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture *Complex Analysis II*
Winter term 2017/2018

Assignment 3B
for the tutorial on *Tuesday, December 5, 4:15 pm* (in Seminar Room 10)

Problem 1.

(i) Consider an automorphism ϕ of the complex upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\},$$

i.e., a biholomorphic mapping $\phi : \mathbb{H} \to \mathbb{H}$. Prove that, if

$$|\phi(z)| \to \infty \quad \text{as} \quad |z| \to \infty,$$

then ϕ must be of the form

$$\phi(z) = \sigma z + \mu \qquad \text{with } \sigma > 0 \text{ and } \mu \in \mathbb{R}.$$

Hint: You may use that $\psi(z) = i\frac{1+z}{1-z}$ defines a biholomorphic mapping $\psi : \mathbb{D} \to \mathbb{H}$; in fact, this mapping ψ is closely related to the biholomorphic mapping from $\mathbb{D}$ to the right half-plane that was used in Problem 2(iii), Assignment 1B. Its inverse mapping $\psi^{-1} : \mathbb{H} \to \mathbb{D}$, which is given by $\psi^{-1}(z) = \frac{z-i}{z+i}$, is called the *Cayley map*.

(ii) Let $K \subset \mathbb{H}$ be a compact $\mathbb{H}$-hull. Prove that if $g_1, g_2 : \mathbb{H}\backslash K \to \mathbb{H}$ are two biholomorphic functions satisfying the hydrodynamic normalization, then necessarily $g_1 = g_2$.

(This proves the uniqueness part of Theorem 4.2 of the lecture.)

Problem 2. Prove Proposition 4.4 of the lecture:

Let $G \subseteq \mathbb{H}$ be a simply connected domain and let $I \subsetneq \mathbb{R}$ be an open intervall with $I \subseteq G^0$. Then, for any $x \in I$, there exists a unique biholomorphic mapping $\phi : G \to \mathbb{H}$ that extends to a homeomorphism $G \cup I \to \mathbb{H} \cup (-1, 1)$ taking x to 0. Moreover, ϕ extends by reflection to a biholomorphic mapping

$$\widetilde{\phi} : \widetilde{G}_I \to \widetilde{\mathbb{H}}_{(-1,1)}.$$

Hint: In order to construct ϕ, establish first the existence of the biholomorphic mapping $\tilde{\phi}$ and then obtain ϕ by restriction to G. On the other hand, for proving uniqueness, start with any biholomorphic function $\psi : G \to \mathbb{H}$ that extends to a homeomorphism $G \cup I \to \mathbb{H} \cup (-1, 1)$; with the help of the Schwarz reflection principle (see Problem 2, Assignment 2B), you may extend ψ to a biholomorphic mapping $\widetilde{\psi} : \widetilde{G}_I \to \widetilde{\mathbb{H}}_{(-1,1)}$; conclude then by using the uniqueness part of the Riemann mapping theorem.