**UNIVERSITÄT DES SAARLANDES**
**FACHRICHTUNG MATHEMATIK**
Dr. Roland Friedrich
Dr. Tobias Mai

**Assignments for the lecture *Complex Analysis II***
Winter term 2017/2018

**Assignment 4A**
for the tutorial on *Tuesday*, ***December 12 (!)***, *4:15 pm* (in Seminar Room 10)

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Throughout the following, we denote by $\Delta_r$ for any $r > 0$ the subset of $\mathbb{C}$ that is given by

$$\Delta_r := \{z \in \mathbb{C} \mid |z| > r\}.$$

**Problem 1.** In the lecture, more precisely in Definition 2.1, we have introduced the two subsets $\Sigma$ and $\Sigma'$ of $\mathcal{O}(\Delta_1)$. Recall that $\Sigma' \subsetneq \Sigma$ and that each $g \in \Sigma$ admits a Laurent expansion around $\infty$ of the form

$$g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}.$$

Now, we define $\Sigma_0 \subset \Sigma$ and $\Sigma_0' \subset \Sigma'$ as the respective subsets consisting of those functions $g$ for which $b_0 = 0$ holds. Let $g \in \Sigma_0$ be given.

(i) Use Grönwall's Area Theorem in order to show that

$$|g(z) - z| \leq 1 \qquad \text{for all } z \in \Delta_2.$$

(ii) Deduce from (i) with the help of Rouché's Theorem that $\Delta_3 \subseteq g(\Delta_2)$.

(iii) Prove that if any $x \in [-3, 3]$ is given, then

$$|g(z) - x| \geq \frac{3}{10}|z| \qquad \text{for all } z \in \Delta_6.$$

**Problem 2.** Let $K \subset \mathbb{H}$ be a compact $\mathbb{H}$-hull.

(i) Suppose that $K \subseteq \overline{D(0, r)}$. Prove that the restriction of $g_K$ to $\mathbb{H} \cap \Delta_r$ extends by reflection to a schlicht holomorphic function $\tilde{g}_K : \Delta_r \to \mathbb{C}$ satisfying

$$\lim_{|z|\to\infty} (\tilde{g}_K(z) - z) = 0.$$

Deduce that the rescaled function $z \mapsto \frac{1}{r}\tilde{g}_K(rz)$ belongs to $\Sigma_0$; use this in order to show that $\Delta_{3r} \subseteq \tilde{g}_K(\Delta_{2r})$ and that for each $x \in [-3r, 3r]$

$$|\tilde{g}_K(z) - x| \geq \frac{3}{10}|z| \qquad \text{for all } z \in \Delta_{6r}.$$

(ii) Prove that

$$|g_K(z) - z| \leq 5\,\mathrm{rad}(K) \qquad \text{for all } z \in \mathbb{H}\backslash K.$$