It is fundamental for the understanding of Brownian motion to be familiar with the nature of (multivariate) Gaussian distributions, in particular of their behaviour under linear transformations. Therefore you should look it up, e.g. in the library or on the Web.

Problem 1. Let $X_1$ and $X_2$ be two Gaussian random variables with $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Assume that $X_1$ and $X_2$ are independent.

1. Show that $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

2. Let $a, b \in \mathbb{R}^+$, and for $X_1$ and $X_2$ the same assumptions as above. What is then $aX_1 + bX_2$?

Problem 2. Invariances of Brownian motion. Let $(B_t)_{t \in \mathbb{R}^+}$ be a standard BM. Show the following properties

1. $(-B_t)_{t \in \mathbb{R}^+}$ is a BM.

2. For $0 < c$, the rescaled Brownian motion $(\frac{1}{c}B_{ct})_{t \in \mathbb{R}^+}$ is a BM.

Problem 3. (Bessel process) Let $B_t$ be an $n$-dim. BM with stochastic Euclidean norm $R_t(\omega) := \|B_t(\omega)\|$. Consider the function $r : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|$. Use the Ito formula in order to determine what $dR_t$ is.

Problem 4. 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-$\sigma$-algebra. Assume that the random variable $X$ is independent of $\mathcal{G}$. Show that $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

2. Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. What is its variation on the interval $[a, b]$?