Proof:

(i) Put $K_{t,t+} := g_{K_t}(K_{t+} \setminus K_t)$. Then, for all $h > 0$ by Theorem 4.8

$$\operatorname{hcap}(K_{t,t+}) \overset{(ii)}{\leq} \operatorname{hcap}(K_{t,t+h}) \overset{(iii)}{\leq} \operatorname{rad}(K_{t,t+h})^2$$

As $h \downarrow 0$, the local growth property yields

$$\operatorname{hcap}(K_{t,t+}) = 0,$$

which by Theorem 4.7 implies $K_{t,t+} = \emptyset$, i.e., $K_{t+} = K_t$.

(ii) If $s < t$ are given, then by Theorem 4.8 (ii)

$$\operatorname{hcap}(K_t) = \operatorname{hcap}(K_s) + \operatorname{hcap}(K_{s,t}).$$

Since $K_s \subsetneq K_t$, we have $K_{s,t} \neq \emptyset$ and so $\operatorname{hcap}(K_{s,t}) > 0$ by Theorem 4.7. Thus

$$\operatorname{hcap}(K_t) > \operatorname{hcap}(K_s),$$

which proves that $t \mapsto \operatorname{hcap}(K_t)$ is strictly increasing. Moreover, by Theorem 4.8 (iii),

$$\operatorname{hcap}(K_t) - \operatorname{hcap}(K_s) = \operatorname{hcap}(K_{s,t}) \leq \operatorname{rad}(K_{s,t})^2.$$

Let $I \subset [0,\infty)$ be a compact interval and let $\varepsilon > 0$. Then

$$\sup_{s \in I} \operatorname{rad}(K_{s,s+h}) \longrightarrow 0 \quad \text{as } h \downarrow 0$$

by the local growth property. Thus, there is a $\delta > 0$

such that

$$\sup_{s\in I} \operatorname{rad}(K_{s,s+h}) < \sqrt{\varepsilon} \quad \forall\ 0<h<\delta.$$

Thus, whenever $s,t \in I$ with $|s-t|<\delta$ are given, then (w.l.o.g. $s<t$)

$$|\operatorname{hcap}(K_t) - \operatorname{hcap}(K_s)| < \varepsilon.$$

Hence, $t \mapsto \operatorname{hcap}(K_t)$ is uniformly continuous on each compact interval $I \subset [0,\infty)$ and therefore continuous on $[0,\infty)$.

(iii) Since $\operatorname{rad}(K_{t,t+h}) \to 0$ as $h \downarrow 0$, we also have that $\operatorname{diam}(\overline{K_{t,t+h}}) \to 0$ as $h \downarrow 0$. Furthermore, the sets $\overline{K_{t,t+h}}$ are compact and decreasing as $h\downarrow 0$. Thus, by Cantor's intersection theorem,

$$\bigcap_{h>0} \overline{K_{t,t+h}} = \{\xi_t\}$$

for some $\xi_t \in \overline{\mathbb{H}}$; in fact, $\xi_t \in \mathbb{R}$, since $\operatorname{rad}(K_{t,t+h}) \to 0$ as $h \downarrow 0$.

In order to prove continuity of the process $(\xi_t)_{t\geq 0}$, we choose $z \in K_{t+2h}\setminus K_{t+h}$ and put

$$w := g_{K_t}(z) \quad \text{and} \quad w' := g_{K_{t+h}}(z).$$

Then

$$w \in K_{t,t+2h} \quad \text{and} \quad w' \in K_{t+h,t+2h},$$

so that

$$|\xi_t - w| \leq 2\,\mathrm{rad}(K_{t,t+2h}),$$

$$|\xi_{t+h} - w'| \leq 2\,\mathrm{rad}(K_{t+h,t+2h}).$$

Furthermore, it holds true that

$$w' = g_{K_{t,t+h}}(w),$$

so that by Theorem 4.9 (i)

$$|w' - w| \leq 5\,\mathrm{rad}(K_{t,t+h}).$$

In summary,

$$|\xi_{t+h} - \xi_t| \longrightarrow 0 \quad \text{as} \quad h \downarrow 0,$$

uniformly on compacts in t.

□

5.3. Remark:

(i) Similarly, one can deal with families of compact $\mathbb{H}$-hulls parametrized over $[0,T)$ for some $0 < T \leq \infty$.

(ii) Let $\tau : [0,T') \to [0,T)$ be a homeomorphism for some $0 < T, T' \leq \infty$ and let $(K_t)_{t\in[0,T)}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property with the Loewner transform $(\mathfrak{Z}_t)_{t\in[0,T)}$. Put

$$K'_t := K_{\tau(t)} \quad \text{and} \quad \mathfrak{Z}'_t := \mathfrak{Z}_{\tau(t)}.$$

Then $(K'_t)_{t\in[0,T')}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property with the Loewner transform $(\mathfrak{Z}'_t)_{t\in[0,T')}$.

(iii) According to Theorem 5.2 (ii), the map

$$t \longmapsto \frac{1}{2}\big(\mathrm{hcap}(K_t) - \mathrm{hcap}(K_0)\big)$$

induces a homeomorphism $\sigma : [0,T) \to [0,T')$ for some $0 < T' \leq \infty$. If we choose $\tau = \sigma^{-1}$, we obtain by (ii) a family $(K'_t)_{t\in[0,T')}$ satisfying

$$\mathrm{hcap}(K'_t) = \mathrm{hcap}(K'_0) + 2t \quad \text{for all } t \in [0,T'). \tag{1}$$

We say that $(K'_t)_{t\in[0,T')}$ is <u>parametrized by half-plane capacity</u>; the factor 2 appears by convention.

<u>5.4. Theorem</u> (Chordal Loewner equation)

Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property and being parametrized by half-plane capacity; let $(\mathfrak{Z}_t)_{t\geq 0}$ be its Loewner transform. We put

$$g_t := g_{K_t} \quad \text{for all } t \geq 0 \quad \text{and}$$
$$T_z := \inf\{t \geq 0 \mid z \in K_t\} \quad \text{for all } z \in \mathbb{H}.$$

Then, for each fixed $z \in \mathbb{H}$, the function

$$[0, T_z) \to \mathbb{C}, \quad t \mapsto g_t(z)$$

is differentiable and satisfies <u>Loewner's differential equation</u>

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \xi_t}.$$

Moreover, if $T_z < \infty$, then

$$g_t(z) - \xi_t \longrightarrow 0 \quad \text{as} \quad t \nearrow T_z.$$

<u>Proof</u>:

Fix $z \in \mathbb{H}$. For each $0 \leq t < T_z$, we put $z_t := g_t(z)$.
Note that for $0 \leq s < t < T_z$ by Theorem 4.8 (ii)

$$\operatorname{hcap}(K_{s,t}) \overset{(1)}{=} \operatorname{hcap}(K_t) - \operatorname{hcap}(K_s) = 2(t-s) \tag{2}$$

and $g_{K_{s,t}}(z_s) = g_{K_{s,t}}(g_{K_s}(z)) = g_{K_t}(z) = z_t$. (3)

Moreover, by Theorem 5.2, we have that $\xi_s \in \overline{K_{s,t}}$, so that

$$K_{s,t} \subseteq \xi_s + 2\operatorname{rad}(K_{s,t})\overline{\mathbb{D}} = \overline{D(\xi_s, 2\operatorname{rad}(K_{s,t}))}. \tag{4}$$

By Theorem 4.9 (i), applied to $K = K_{s,t}$ and $z = z_s$, we get

$$|z_t - z_s| \overset{(3)}{=} |g_{K_{s,t}}(z_s) - z_s| \leq 5\operatorname{rad}(K_{s,t}), \tag{5}$$

and by Theorem 4.9 (ii), for $\xi = \xi_s$ and $r = 2\,\mathrm{rad}(K_{s,t})$,

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - \xi_s} \right| \overset{(2),(3)}{=} \left| g_{K_{s,t}}(z_s) - z_s - \frac{\mathrm{hcap}(K_{s,t})}{z_s - \xi_s} \right|$$

$$\overset{(4)}{\leq} \frac{C\, 2\,\mathrm{rad}(K_{s,t})\,\mathrm{hcap}(K_{s,t})}{|z_s - \xi_s|^2}$$

$$\overset{(2)}{=} \frac{4C\,\mathrm{rad}(K_{s,t})}{|z_s - \xi_s|^2}(t-s) \qquad (6)$$

holds whenever $|z_s - \xi_s| \geq 6r = 12\,\mathrm{rad}(K_{s,t})$.

By (5) and the local growth property, we see that

$$[0, T_z) \to \mathbb{C}, \quad t \mapsto z_t = g_t(z)$$

is continuous. We claim that

$$\lim_{t \downarrow s} \frac{z_t - z_s}{t-s} = \frac{2}{z_s - \xi_s}, \qquad (7)$$

from which the differentiability of $t \mapsto g_t(z)$ with

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \xi_t}$$

follows (!). For proving (7), we fix $0 \leq s < T_z$ and set

$$\varepsilon_0 := |z_s - \xi_s| > 0.$$

By the local growth property, if $\varepsilon > 0$ is given, we find $\delta > 0$ s.t.

$$\mathrm{rad}(K_{s,t}) < \min\left\{ \frac{\varepsilon_0}{12}, \frac{\varepsilon_0^2}{4C} \cdot \varepsilon \right\} \quad \forall t \in (s, s+\delta).$$

Hence, $|z_s - \xi_s| = \varepsilon_0 > 12\,\mathrm{rad}(K_{s,t})$ if $t \in (s, s+\delta)$, so that by (6)

$$\left| \frac{z_t - z_s}{t-s} - \frac{2}{z_s - \xi_s} \right| \leq \frac{4C}{\varepsilon_0^2} \operatorname{rad}(K_{s,t}) < \varepsilon$$

for all $t \in (s, s+\delta)$. This shows (7).

Finally, if $T_z < \infty$, then $z \in K_t \setminus K_s$ for $0 \leq s < T_z < t$. Thus, $z_s = g_s(z) \in g_{K_s}(K_t \setminus K_s) = K_{s,t}$, so that

$$|z_s - \xi_s| \leq 2 \operatorname{rad}(K_{s,t}).$$

Choose $\delta_0 > 0$, such that $I := [T_z - \delta_0, T_z + \delta_0] \subseteq (0, \infty)$. If $\varepsilon > 0$ is given, we find by the local growth property some $\delta_1 > 0$, such that

$$\sup_{s \in I} \operatorname{rad}(K_{s,s+h}) < \frac{\varepsilon}{2} \quad \text{whenever } 0 < h < \delta_1.$$

Put $\delta := \min\{\delta_0, \frac{1}{2}\delta_1\}$. Then, for $s \in (T_z - \delta, T_z) \subset I$,

$$t := T_z + (T_z - s) = 2T_z - s \in (T_z, T_z + \delta)$$

and hence $h := t - s = 2(T_z - s) < \delta_1$. Thus

$$|z_s - \xi_s| \leq 2 \operatorname{rad}(K_{s,t}) < \varepsilon.$$

Thus, $g_s(z) - \xi_s = z_s - \xi_s \longrightarrow 0$ as $s \nearrow T_z$.

$\square$

5.5. Remark

In the situation of Theorem 5.4 put $f_t := g_t^{-1}$ for all $t \geq 0$. Then, since

$$0 = \frac{\partial}{\partial t} \underbrace{(g \circ f)}_{= \mathrm{id}_{\mathbb{H}}} = \frac{\partial g}{\partial t} \circ f + \frac{\partial g}{\partial z} \circ f \cdot \frac{\partial f}{\partial t}$$

and

$$1 = \frac{\partial}{\partial z}(g \circ f) = \frac{\partial g}{\partial z} \circ f \cdot \frac{\partial f}{\partial z},$$

we see that Loewner's differential equation yields (and is in fact equivalent to)

$$\frac{\partial}{\partial t} f_t(z) = - \frac{\partial}{\partial z} f_t(z) \cdot \frac{2}{z - \xi_t}, \qquad z \in \mathbb{H}.$$

5.6. Examples:

(i) Put $K_t := t\,\overline{\mathbb{D}} \cap \mathbb{H}$ for $t \geq 0$. Then, if $t > 0$,

$$g_{K_t}(z) = t\, g_{K_1}\left(\frac{z}{t}\right) = t\left(\frac{z}{t} + \frac{t}{z}\right) = z + \frac{t^2}{z}$$

by Theorem 4.8 (i) and Example 4.5 (i); thus

$$g_{K_t}(z) = z + \frac{t^2}{z} \quad \text{for all } t \geq 0.$$

Since, for $t < r \leq t + h$,

$$g_{K_t}(re^{i\theta}) = r\left(1 + \left(\frac{t}{r}\right)^2\right)\cos(\theta) + i r\left(1 - \left(\frac{t}{r}\right)^2\right)\sin(\theta),$$

we see that (uniformly on compacts in t)

$$\operatorname{rad}(K_{t,t+h}) = (t+h)\left(1 + \left(\frac{t}{t+h}\right)^2\right) \xrightarrow{h \downarrow 0} 2t.$$

Thus, $(K_t)_{t \geq 0}$ is increasing but does not have the local growth property.

(ii) Put $K_t := (0, 2\sqrt{t}\, i]$ for $t \geq 0$. Then, if $t > 0$,

$$g_{K_t}(z) = 2\sqrt{t}\, g_{K_1}\left(\frac{z}{2\sqrt{t}}\right) = 2\sqrt{t}\sqrt{\frac{z^2}{4t} + 1} = \sqrt{z^2 + 4t}$$

by Theorem 4.8 (i) and Example 4.5 (ii); thus

$$g_{K_t}(z) = \sqrt{z^2+4t} \quad \text{for all } t \geq 0.$$

Moreover, for $t>0$,

$$\operatorname{hcap}(K_t) = (2\sqrt{t})^2 \underbrace{\operatorname{hcap}(K_1)}_{=1/2} = 2t$$

and $\operatorname{hcap}(K_0) = 0$, so that $(K_t)_{t\geq 0}$ is parametrized by half-plane capacity.

Since, for $t < r \leq t+h$,

$$g_{K_t}(2\sqrt{r}\,i) = \sqrt{(2\sqrt{r}\,i)^2 + 4t} = 2\sqrt{r-t}\,i,$$

we see that (uniformly on compacts in t)

$$\operatorname{rad}(K_{t,t+h}) = \operatorname{rad}((0, 2\sqrt{h}\,i]) = 2h \xrightarrow{h\downarrow 0} 0.$$

Thus, $(K_t)_{t\geq 0}$ is increasing and has the local growth property. We check that

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{\sqrt{z^2+4t}} = \frac{2}{g_t(z)},$$

which corresponds to $\xi_t = 0$ for all $t \geq 0$; indeed

$$\bigcap_{h>0} \overline{K_{t,t+h}} = \bigcap_{h>0} [0, 2\sqrt{h}\,i] = \{0\}.$$

5.7. Remark:

(i) Let $\gamma : [0,\infty) \to \mathbb{C}$ be continuous and injective with $\gamma(0) \in \mathbb{R}$ and $\gamma((0,\infty)) \subset \mathbb{H}$. For $t \geq 0$, put $K_t := \gamma((0,t])$, where $K_0 = \emptyset$. Then $(K_t)_{t\geq 0}$ yields an increasing family of compact $\mathbb{H}$-hulls.

One can show that, for some universal constant $c > 0$,

$$\operatorname{diam}\left(g_t\left(\gamma((t,t+h])\right)\right) \le c\sqrt{\operatorname{diam}\left(\gamma((0,T])\right)\operatorname{osc}(\gamma,h,T)},$$

where

$$\operatorname{osc}(\gamma,h,T) := \sup_{\substack{s,t\in[0,T]:\\ |s-t|<h}} |\gamma(s)-\gamma(t)|$$

and $\operatorname{diam}(K) := \sup_{z,w\in K} |z-w|$.

Thus, $(K_t)_{t\ge 0}$ has the local growth property. (Note that $\operatorname{rad}(K) \le \operatorname{diam}(K)$ for each compact $\mathbb{H}$-hull, as $|z-\xi| \le \operatorname{diam}(K)$ for all $z \in K$ if $\xi \in \overline{K}\cap\mathbb{R}$.)

Moreover, we have in this case, for each $t \ge 0$, that

$$g_t(z) \longrightarrow \xi_t \quad \text{as} \quad z \to \gamma(t) \ \text{ in } \mathbb{H}\setminus K_t.$$

(ii) Conversely, let any continuous <u>driving function</u> $\xi : [0,\infty) \to \mathbb{R}$ be given. For fixed $z \in \mathbb{H}$, let

$$[0,T_z) \longrightarrow \mathbb{H}, \quad t \longmapsto g_t(z)$$

be the <u>maximal solution</u> of the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z)-\xi_t}, \qquad g_0(z) = z,$$

with *lifetime* $T_z \in (0,\infty]$. Then, for any $t \geq 0$, we obtain a compact $\mathbb{H}$-hull

$$K_t := \{ z \in \mathbb{H} \mid T_z \leq t \}$$

and we have that $g_t = g_{K_t}$. Moreover, $(K_t)_{t \geq 0}$ is an increasing family of compact $\mathbb{H}$-hulls that has the local growth property and that is parametrized by half-plane capacity; its Loewner transform is $(\xi_t)_{t\geq 0}$.

We call $(g_t)_{t\geq 0}$ the *Loewner chain* associated to $(\xi_t)_{t \geq 0}$.

(iii) We say that a Loewner chain $(g_t)_{t\geq 0}$ is *generated by a curve*, if we find a continuous function $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ with $\gamma(0) \in \mathbb{R}$, such that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0,t])$ for each $t \geq 0$; it can be shown that this is the case, if $\overline{K_t}$ is locally connected for each $t \geq 0$: a closed set $K \subset \mathbb{C}$ is said to be *locally connected*, if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall z, w \in K : \ |z-w| < \delta \Rightarrow \exists K' \subset K \text{ connected: } z, w \in K', \operatorname{diam}(K') < \varepsilon$$

Note that γ is not necessarily injective. A result of Marshall and Rhode (2005) says that $C > 0$ exists, such that $(g_t)_{t\geq 0}$ is generated by a simple curve, if $|\xi_t - \xi_s| \leq C |t-s|^{1/2}$ for all $s,t \geq 0$.