and

$$z\left(g_K(z)-z\right) \longrightarrow \left(\frac{b_2}{b_1}\right)^2 - \frac{b_3}{b_1} \in \mathbb{R} \qquad \text{as } |z|\to\infty.$$

Uniqueness is left as an exercise. $\square$

4.5. Example:

(i) If $K := \overline{\mathbb{D}} \cap \mathbb{H}$, then

$$g_K(z) = z + \frac{1}{z} \qquad \text{and} \qquad \operatorname{hcap}(K) = 1.$$

(ii) If $K := (0,i] = \{iy \mid y \in (0,1]\}$, then

$$g_K(z) = \sqrt{z^2+1} = z + \frac{1}{2z} + O(|z|^{-2})$$

and thus $\operatorname{hcap}(K) = \frac{1}{2}$.

In fact, we always have that $\operatorname{hcap}(K) \geq 0$. To see this, we need some further tools.

4.6. Remark:

(i) A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ with

$$\operatorname{Im}(f(z)) \geq 0 \qquad \text{for all } z \in \mathbb{H}$$

is called a Pick function (or Nevanlinna function).

(ii) A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a Pick function if and only if it can be written as

$$f(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{x}{1+x^2}\right) d\mu(x)$$

for some $\alpha \in \mathbb{R}$, $\beta \geq 0$ and some Borel measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{1}{1+x^2}\, d\mu(x) < \infty .$$

(iii) We denote by $\mathcal{P}$ the set of all Pick functions that are of the special form

$$f(z) = z + \int_{\mathbb{R}} \frac{1}{x-z}\, d\mu(x)$$

for some finite Borel measure μ on $\mathbb{R}$.

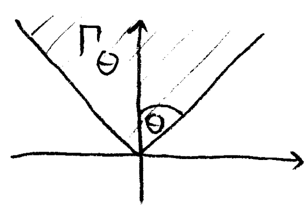
(iv) It can be shown that a Pick function f belongs to $\mathcal{P}$ if and only if

- $$f(z) - z \longrightarrow 0 \quad \text{as} \quad z \to \infty$$

  in the sense that

  $$\forall \eta > 0 \ \forall \varepsilon > 0 \ \exists M > 0 : \ |f(z) - z| < \varepsilon \ \text{for } z \in \mathbb{H} \text{ with } \operatorname{Im}(z) > \eta,\ |z| > M,$$

- $$z\,(f(z) - z) \longrightarrow c \quad \text{as} \quad z \to \infty$$

  in the sense of angular limits, i.e.,

  $$\forall \Theta \in (0, \tfrac{\pi}{2})\ \forall \varepsilon > 0\ \exists M > 0 : |z(f(z) - z) - c| < \varepsilon \text{ for } z \in \Gamma_\Theta, |z| > M,$$

  where $\Gamma_\Theta := \{ z \in \mathbb{H} \mid |\operatorname{Re}(z)| \leq \tan(\Theta) \operatorname{Im}(z) \}$.

In this case $c = -\mu(\mathbb{R})$. Moreover, there exists a reflection symmetric $\tilde{f} \in \mathcal{O}(\mathbb{C} \setminus \overline{D(0,r)})$ with $\tilde{f}|_{\mathbb{H} \setminus \overline{D(0,r)}} = f|_{\mathbb{H} \setminus \overline{D(0,r)}}$ for some $r > 0$, if and only if $\operatorname{supp}(\mu) \subseteq [-r, r]$.

4.7. Theorem:

For each compact $\mathbb{H}$-hull K, we have that

$$\operatorname{hcap}(K) \geq 0,$$

with equality if and only if $K = \emptyset$.

Proof:

Consider $f_K := g_K^{-1} : \mathbb{H} \to \mathbb{H} \setminus K$, which is then a Pick function. In fact, since

$$g_K(z) - z \longrightarrow 0 \quad \text{as } |z| \to \infty$$

and

$$z(g_K(z) - z) \longrightarrow \operatorname{hcap}(K) \quad \text{as } |z| \to \infty,$$

we have that

$$f_K(z) - z \longrightarrow 0 \quad \text{as } |z| \to \infty$$

and

$$z(f_K(z) - z) \longrightarrow -\operatorname{hcap}(K) \quad \text{as } |z| \to \infty,$$

and hence, according to Remark 4.6 (iv), $f_K \in \mathcal{P}$ with

$$-\operatorname{hcap}(K) = c = -\mu(\mathbb{R}),$$

i.e., $\operatorname{hcap}(K) = \mu(\mathbb{R}) \geq 0$. In particular,

$$\operatorname{hcap}(K) = 0 \iff \mu = 0 \iff f_K = \mathrm{id} \iff K = \emptyset.$$

$\square$

4.8. Theorem:

(i) Let K be a compact $\mathbb{H}$-hull. For any $r > 0$ and $x \in \mathbb{R}$, we put

$$rK := \{ rz \mid z \in K \} \quad \text{and} \quad K + x := \{ z + x \mid z \in K \}$$

Then rK and $K + x$ are compact $\mathbb{H}$-hulls with

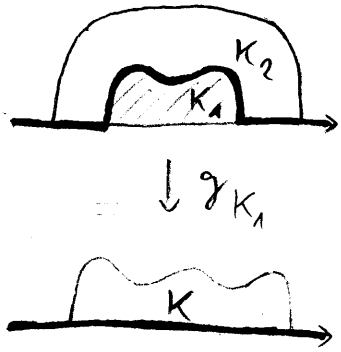
$$g_{rK}(z) = r\, g_K\left(\frac{z}{r}\right) \quad \text{and} \quad g_{K+x}(z) = g_K(z-x) + x.$$

Thus

$$\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K) \quad \text{and} \quad \operatorname{hcap}(K+x) = \operatorname{hcap}(K).$$

(ii) Let $K_1 \subseteq K_2$ be compact $\mathbb{H}$-hulls. Then

$$K := g_{K_1}(K_2 \setminus K_1)$$



is a compact $\mathbb{H}$-hull and $g_{K_2} = g_K \circ g_{K_1}$. Thus

$$\operatorname{hcap}(K_2) = \operatorname{hcap}(K_1) + \operatorname{hcap}(K).$$

In particular, $\operatorname{hcap}(K_2) \geq \operatorname{hcap}(K_1)$.

(iii) Let K be a compact $\mathbb{H}$-hull. We define

$$\operatorname{rad}(K) := \min \{ r \geq 0 \mid \exists \xi \in \mathbb{R}\ \forall z \in K : |z - \xi| \leq r \}$$

Then $K \subseteq \operatorname{rad}(K)(\overline{\mathbb{D}} \cap \mathbb{H} + \xi)$ for some $\xi \in \mathbb{R}$, so that

$$\operatorname{hcap}(K) \overset{(ii)}{\leq} \operatorname{hcap}(\operatorname{rad}(K)(\overline{\mathbb{D}} \cap \mathbb{H} + \xi))$$

$$\overset{(i)}{=} \operatorname{rad}(K)^2 \underbrace{\operatorname{hcap}(\overline{\mathbb{D}} \cap \mathbb{H})}_{= 1, \text{ by Example 4.5(i)}} = \operatorname{rad}(K)^2$$

4.9. Theorem:

Let K be a compact $\mathbb{H}$-hull.

(i) We have the continuity estimate

$$|g_K(z) - z| \leq 3\,\mathrm{rad}(K) \qquad \forall z \in \mathbb{H} \setminus K$$

(where the constant 3 is optimal).

(ii) There is an absolute constant $C > 0$ for which the differentiability estimate holds: If $r \in (0, \infty)$ and $\xi \in \mathbb{R}$ are such that $K \subseteq \overline{D(\xi, r)}$ holds, then

$$\left| g_K(z) - z - \frac{\mathrm{hcap}(K)}{z - \xi} \right| \leq \frac{C\, r\, \mathrm{hcap}(K)}{|z - \xi|^2}$$

for all $z \in \mathbb{H} \setminus \overline{D(\xi, 2r)}$.

In the following, we put $\Delta_r := \{ z \in \mathbb{C} \mid |z| > r \}$ for $r > 0$.

Proof

(i) Here, we only need the weaker version

$$|g_K(z) - z| \leq 5\,\mathrm{rad}(K) \qquad \forall z \in \mathbb{H} \setminus K;$$

this is Problem 2(ii), Assignment 4A.

(ii) By translating K if necessary, we may assume that $\xi = 0$.

Thus, it suffices to prove that

$$\left| g_K(z) - z - \frac{\mathrm{hcap}(K)}{z} \right| \leq \frac{C r \, \mathrm{hcap}(K)}{|z|^2}$$

for all $z \in \mathbb{H} \setminus \overline{D(0,2r)}$. We need this only for $z \in \mathbb{H} \setminus \overline{D(0,6r)}$. Consider $f_K := g_K^{-1} \in \mathcal{P}$. Then, for $z \in \mathbb{H}$,

$$f_K(z) = z + \int_{\mathbb{R}} \frac{1}{x-z} \, d\mu(x),$$

for some measure μ with $\mu(\mathbb{R}) = \mathrm{hcap}(K)$. By Problem 2 (i), Assignment 4A, g_K extends ~~to~~ by reflection to a schlicht holomorphic function

$$\tilde{g}_K : \Delta_r \to \mathbb{C} \quad \text{with} \quad \Delta_{3r} \subseteq \tilde{g}_K(\Delta_{2r}).$$

Thus, f_K admits the reflection symmetric extension

$$\tilde{f}_K := \tilde{g}_K^{-1}|_{\Delta_{3r}} : \Delta_{3r} \to \mathbb{C},$$

so that Remark 4.6 gives $\mathrm{supp}(\mu) \subseteq [-3r, 3r]$.

Now, we see that for all $z \in \mathbb{H} \setminus K$

$$z = f_K(g_K(z)) = g_K(z) + \int_{\mathbb{R}} \frac{1}{x - g_K(z)} \, d\mu(x)$$

$$\Longrightarrow \quad g_K(z) = z - \int_{\mathbb{R}} \frac{1}{x - g_K(z)} \, d\mu(x)$$

$$= z + \frac{\mathrm{hcap}(K)}{z} - \int_{[-3r,3r]} \left(\frac{1}{x - g_K(z)} + \frac{1}{z} \right) d\mu(x)$$

$$= z + \frac{\mathrm{hcap}(K)}{z} - \int_{[-3r,3r]} \frac{z - g_K(z) + x}{(x - g_K(z))\, z}\, d\mu(x)$$

Thus, by Problem 2(i), Assignment 4 A, and part (i)

$$\left| g_K(z) - z - \frac{\mathrm{hcap}(K)}{z} \right| \leq \mathrm{hcap}(K) \cdot \frac{8r}{\frac{3}{10}|z|^2}$$

$$= \frac{C\, r\, \mathrm{hcap}(K)}{|z|^2}$$

for all $z \in \mathbb{H} \setminus \overline{D(0,6r)}$, with $C := \frac{80}{3}$.

$\square$

5. The chordal Loewner theory

5.1. Definition:

Let $(K_t)_{t\geq 0}$ be a family of compact $\mathbb{H}$-hulls. We say that

(i) $(K_t)_{t\geq 0}$ is increasing, if $K_s \subsetneq K_t$ for all $s<t$.

(ii) $(K_t)_{t\geq 0}$ has the local growth property, if

$$\mathrm{rad}(K_{t,t+h}) \longrightarrow 0$$

uniformly on compacts in t as $h \downarrow 0$.

Here, we use the notation

$$K_{s,t} := g_{K_s}(K_t \setminus K_s) \quad \text{for all } s<t.$$

5.2. Theorem:

Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}$-hulls that has the local growth property.

(i) For all $t\geq 0$, we have that

$$K_{t+} := \bigcap_{s>t} K_s = K_t.$$

(ii) The map $t \mapsto \mathrm{hcap}(K_t)$ is continuous and strictly increasing on $[0,\infty)$.

(iii) For each $t\geq 0$, there is a unique $\xi_t \in \mathbb{R}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h>0$. Moreover, the process $(\xi_t)_{t\geq 0}$, the Loewner transform of $(K_t)_{t\geq 0}$, is continuous.