Then

$$|a_2| \leq 2,$$

with equality if and only if f is a rotation of the Koebe function k_0, i.e. $f = k_\Theta$ for some $\Theta \in [0, 2\pi)$.

Recall from Problem 2, Assignment 1B, that

$$k_\Theta : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{(1 - ze^{i\Theta})^2} \quad \text{and}$$

$$k_0 : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{(1-z)^2}.$$

Proof:

Consider the square root transform $h \in \mathcal{S}$ of f (see Problem 1 (ii), Assignment 1B); write

$$h(z) = z + \sum_{n=1}^{\infty} \alpha_{2n+1} z^{2n+1}.$$

Now, define $g \in \Sigma'$ by

$$g : \Delta \to \mathbb{C}, \quad \zeta \mapsto \frac{1}{h(\frac{1}{\zeta})};$$

we write

$$g(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}.$$

By Problem 1 (iii), Assignment 1B, we have

$$a_2 = 2\alpha_3,$$

and by Remark 2.2., we have

$$b_0 = 0 \quad \text{and} \quad b_1 = -\alpha_3.$$

Thus
$$b_1 = -\frac{a_2}{2}.$$

Finally, Corollary 2.5 gives $|b_1| \leq 1$; thus $|a_2| \leq 2$, with equality if and only if (since $b_0 = 0$)

$$g(\zeta) = \zeta + \frac{b_1}{\zeta}, \quad b_1 = -e^{i\theta}$$

$$\Rightarrow h(z) = \frac{1}{g(\frac{1}{z})} = \frac{1}{\frac{1}{z} - e^{i\theta} z} = \frac{z}{1 - e^{i\theta} z^2}$$

$$\Rightarrow f(z^2) = h(z)^2 = \frac{z^2}{(1 - e^{i\theta} z^2)^2}$$

$$\Rightarrow f(z) = \frac{z}{(1 - e^{i\theta} z)^2} = k_\theta(z).$$

$\square$

2.7. Theorem:

(i) Let $\phi : \mathbb{D} \to \mathbb{D}$ be an automorphism of $\mathbb{D}$. Then, for any schlicht $f : \mathbb{D} \to \mathbb{C}$,

$$g : \mathbb{D} \to \mathbb{C}, \quad z \longmapsto \frac{f(\phi(z)) - f(\phi(0))}{f'(\phi(0))\, \phi'(0)}$$

is a function in $\mathcal{S}$ (Koebe transform of f).

(ii) If $f \in \mathcal{S}$ and $w \in \mathbb{C} \setminus f(\mathbb{D})$ are given, then

$$g : \mathbb{D} \to \mathbb{C}, \quad z \longmapsto \frac{w f(z)}{w - f(z)}$$

belongs to $\mathcal{S}$.

Proof: direct computation! $\square$

2.8. Theorem (Koebe $\frac{1}{4}$-Theorem):

Let $f \in \mathcal{S}$ be given. Then

$$D(0, \tfrac{1}{4}) \subseteq f(\mathbb{D}).$$

There is a point $w \in \mathbb{C} \setminus f(\mathbb{D})$ with $|w| = \frac{1}{4}$ if and only if $f = k_\Theta$ for some $\Theta \in [0, 2\pi)$; more precisely, we have that $w = -\frac{1}{4} e^{-i\Theta}$.

Proof:

Take $w \in \mathbb{C} \setminus f(\mathbb{D}) \neq \emptyset$ and consider

$$g: \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{w f(z)}{w - f(z)},$$

which belongs then to $\mathcal{S}$ due to Theorem 2.7 (ii). One easily sees that

$$g(z) = z + \left(a_2 + \frac{1}{w}\right) z^2 + \ldots .$$

Thus, by Theorem 2.6, both

$$\left| a_2 + \frac{1}{w} \right| \leq 2 \quad \text{and} \quad |a_2| \leq 2.$$

Hence, $\frac{1}{|w|} \leq |a_2 + \frac{1}{w}| + |a_2| \leq 4$, i.e. $|w| \geq \frac{1}{4}$, with equality if and only if $a_2 = 2e^{i\Theta}$ and $\frac{1}{w} = -4e^{i\Theta}$ for some $\Theta \in [0, 2\pi)$; in particular, $f = k_\Theta$. $\square$

2.9. Proposition:

If $f \in \mathcal{S}$, then for all $z \in \mathbb{D}$

$$\left| \frac{f''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \le \frac{4|z|}{1-|z|^2} .$$

Proof:

Fix $a \in \mathbb{D}$ and consider the automorphism

$$\phi = \phi_{-a} : \mathbb{D} \to \mathbb{D}, \; z \mapsto \frac{z+a}{1+\bar{a}z},$$

which satisfies $a = \phi(0)$ and $\phi'(0) = 1 - |a|^2$.

According to Theorem 2.7 (i), the Koebe transform

$$g : \mathbb{D} \to \mathbb{C}, \; z \mapsto \frac{f(\phi(z)) - f(a)}{(1-|a|^2) f'(a)}$$

belongs to $\mathcal{S}$; we write

$$g(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n , \quad z \in \mathbb{D},$$

so that, by Theorem 2.6, $\frac{1}{2}|g''(0)| = |\alpha_2| \le 2$. Since

$$\phi'(0) = 1 - |a|^2 \quad \text{and} \quad \phi''(0) = -2a(1-|a|^2),$$

we obtain that

$$g''(0) = \frac{f''(a)}{f'(a)} \cdot (1-|a|^2) - 2\bar{a},$$

so that

$$\left| \frac{f''(a)}{f'(a)} \cdot (1-|a|^2) - 2\bar{a} \right| \le 4.$$

Multiplying with $\frac{|a|}{1-|a|^2}$ yields the asserted inequality.

$\square$

2.10. Theorem (Distortion Theorem)

If $f \in \mathcal{S}$, then for all $z \in \mathbb{D}$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Proof:

By Theorem 1.3., the function $f' : \mathbb{D} \to \mathbb{C}$ is everywhere non-zero; thus, there is a holomorphic function $F : \mathbb{D} \to \mathbb{C}$, such that $f'(z) = \exp(F(z))$ for all $z \in \mathbb{D}$; since $f'(0) = 1$, we may assume $F(0) = 0$. Given $z = re^{i\theta} \in \mathbb{D}$, we obtain

$$\frac{\partial}{\partial r} F(re^{i\theta}) = F'(re^{i\theta})\, e^{i\theta}$$

$$\Longrightarrow \quad r \cdot \frac{\partial}{\partial r} F(re^{i\theta}) = z \cdot F'(z) = z \cdot \frac{f''(z)}{f(z)}.$$

Now, from Proposition 2.9., we infer that

$$\left| r \frac{\partial}{\partial r} F(re^{i\theta}) - \frac{2r^2}{1-r^2} \right| \leq \frac{4r}{1-r^2}$$

and hence (since $-|w| \leq \mathrm{Re}(w) \leq |w|$ for all $w \in \mathbb{C}$)

$$-\frac{4}{1-r^2} \leq \mathrm{Re}\left(\frac{\partial}{\partial r} F(re^{i\theta})\right) - \frac{2r}{1-r^2} \leq \frac{4}{1-r^2}.$$

Now, note that $|f'(z)| = \exp(\mathrm{Re}(F(z)))$ and hence $\mathrm{Re}(F(z)) = \log|f'(z)|$, so that

$$\mathrm{Re}\left(\frac{\partial}{\partial r} F(re^{i\theta})\right) = \frac{\partial}{\partial r}\mathrm{Re}(F(re^{i\theta})) = \frac{\partial}{\partial r}\log|f'(re^{i\theta})|.$$

Thus:

$$-\frac{4}{1-r^2} \le \frac{\partial}{\partial r}\log|f'(re^{i\theta})| - \frac{2r}{1-r^2} \le \frac{4}{1-r^2}$$

$$\Longrightarrow \quad \frac{2r-4}{1-r^2} \le \frac{\partial}{\partial r}\log|f'(re^{i\theta})| \le \frac{2r+4}{1-r^2}$$

Since $\log|f'(0)| = 0$, integration over $[0,R]$ for any $0 < R < 1$ gives

$$\log\left(\frac{1-R}{(1+R)^3}\right) \le \log|f'(Re^{i\theta})| \le \log\left(\frac{1+R}{(1-R)^3}\right)$$

$$\Longrightarrow \quad \frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}$$

for any point $z = Re^{i\theta} \in \mathbb{D}\setminus\{0\}$
(z = 0 is trivial) □

2.11. Theorem (Growth Theorem):

If $f \in \mathcal{S}$, then for all $z \in \mathbb{D}$

$$\frac{|z|}{(1+|z|)^2} \overset{②}{\le} |f(z)| \overset{①}{\le} \frac{|z|}{(1-|z|)^2}.$$

Proof:

① Let $z = Re^{i\theta} \in \mathbb{D}$ be given. Since

$$f(z) = \int_0^R f'(re^{i\theta})\, e^{i\theta}\, dr,$$

we get from Theorem 2.10

$$|f(z)| \le \int_0^R |f'(re^{i\theta})|\, dr$$

$$\le \int_0^R \frac{1+r}{(1-r)^3}\, dr = \frac{R}{(1-R)^2} = \frac{|z|}{(1-|z|)^2}$$

② For $z \in \mathbb{D}$, we consider two cases:

Case 1: $|f(z)| \ge \frac{1}{4}$

Since $h : [0,1] \to \mathbb{R},\ r \mapsto \frac{r}{(1+r)^2}$ is monotonic increasing, we get that

$$\frac{|z|}{(1+|z|)^2} = h(|z|) \le h(1) = \frac{1}{4} \le |f(z)|.$$

Case 2: $|f(z)| < \frac{1}{4}$

By Theorem 2.8, $D(0, \frac{1}{4}) \subseteq f(\mathbb{D})$; thus

$$\{ t\, f(z) \mid 0 \le t \le 1 \} \subset f(\mathbb{D}).$$

Consider the smooth path

$$\gamma : [0,1] \to \mathbb{D},\quad t \mapsto f^{-1}(t\, f(z)).$$

Then $f \circ \gamma = \gamma_{0 \to f(z)}$, hence $L(f \circ \gamma) = |f(z)|$.

Now, choose smooth functions

$r : [0,1] \to [0,1)$ and $\theta : [0,1] \to \mathbb{R}$,

such that $\gamma(t) = r(t) e^{i\theta(t)}$ for all $t \in [0,1]$.

Note that

$$\gamma'(t) = r'(t) e^{i\theta(t)} + r(t)\, i\theta'(t)\, e^{i\theta(t)}$$

$$\Rightarrow \quad |\gamma'(t)|^2 = |r'(t) + i\, r(t)\,\theta'(t)|^2$$

$$= r'(t)^2 + r(t)^2 \theta'(t)^2 \geq r'(t)^2$$

and hence $|\gamma'(t)| \geq |r'(t)| \geq r'(t)$. Thus, we get

$$
\begin{aligned}
|f(z)| &= L(f\circ\gamma) \\
&= \int_0^1 |(f\circ\gamma)'(t)|\, dt \\
&= \int_0^1 |f'(\gamma(t))|\, |\gamma'(t)|\, dt \\
&\overset{\text{Theorem 2.10}}{\geq} \int_0^1 \frac{1-|\gamma(t)|}{(1+|\gamma(t)|)^3}\, |\gamma'(t)|\, dt \\
&\geq \int_0^1 \frac{1-r(t)}{(1+r(t))^3}\, r'(t)\, dt \\
&= \int_{r(0)}^{r(1)} \frac{1-s}{(1+s)^3}\, ds \\
&= \int_0^{|z|} \frac{1-s}{(1+s)^3}\, ds \\
&= \frac{|z|}{(1+|z|)^2}
\end{aligned}
$$

(Note:
$\gamma(0) = 0 \Rightarrow r(0) = 0$
$\gamma(1) = z \Rightarrow r(1) = |z|$)

$\square$