2.12 Remark:

(i) By careful examination of the proofs, one sees that equality in one of the four inequalities of Theorem 2.10 and 2.11 at some $z \in \mathbb{D} \setminus \{0\}$ enforces that f is a rotation of the Koebe function.

(ii) Using Theorem 2.11 and Hurwitz's theorem, one sees that $\mathcal{S}$ is a compact subset of the complete metric space $(\mathcal{O}(\mathbb{D}), d)$, where d denotes the metric of compact convergence.

(iii) The coefficient functionals

$$a_n : (\mathcal{O}(\mathbb{D}), d) \to \mathbb{C}, \quad f \mapsto a_n(f)$$

are continuous, since

$$a_n(f) = \frac{1}{2\pi i} \int_{\gamma_{0,r,\sigma}} \frac{f(z)}{z^{n+1}} \, dz$$

$$\Rightarrow \quad |a_n(f)| \leq \frac{1}{r^n} \|f\|_{\partial D(0,r)}, \quad r \in (0,1).$$

Thus, by (ii): $\sup\limits_{f \in \mathcal{S}} |a_n(f)| < \infty \quad \forall n \geq 2.$

(iv) In fact, one can show that (Littlewood, 1925)

$$|a_n| < e n \qquad \forall n \geq 2$$

for all $f \in \mathcal{S}$. Thus, the Bieberbach conjecture gives the correct order $|a_n| = O(n)$.

(v) Even better (Hayman, 1955) for each $f \in \mathcal{S}$

$$\alpha := \lim_{n\to\infty} \frac{|a_n|}{n}$$

exists and $\alpha \leq 1$, with equality if and only if f is a rotation of the Koebe function.

The Hayman index α can be computed by

$$\lim_{r \nearrow 1} \frac{1-r^2}{r} \max_{\theta \in [0,2\pi]} |f(re^{i\theta})| = \alpha .$$

(vi) Another interesting application of Prop. 2.9 is the following:

For all $0 < r \leq r_0 := 2-\sqrt{3}$, the disc $D(0,r)$ is mapped by each $f \in \mathcal{S}$ to a convex domain; this fails for $r > r_0$.

3. The Carathéodory kernel theorem

Let us consider the following situation:

$$\left.\begin{array}{l} \bullet \text{ Let } (G_n)_{n=1}^{\infty} \text{ be a sequence of} \\ \text{simply connected domains } G_n \subsetneq \mathbb{C} \\ \text{with } 0 \in G_n. \\ \bullet \text{ Let } (f_n)_{n=1}^{\infty} \text{ be the corresponding} \\ \text{sequence of biholomorphic mappings} \\ f_n : \mathbb{D} \to G_n \text{ satisfying } f_n(0) = 0 \\ \text{and } f_n'(0) > 0 \text{ (see Theorem 1.5)} \end{array}\right\} \quad (*)$$

Our goal is to introduce a suitable notion of convergence for $(G_n)_{n=1}^{\infty}$, such that

$$\boxed{\begin{array}{l} f_n \to f \\ \text{compactly on } \mathbb{D} \end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{l} G_n \to G \\ \text{with } G = f(\mathbb{D}) \end{array}}$$

3.1. Definition:

Let $(G_n)_{n=1}^{\infty}$ be any sequence of domains in $\mathbb{C}$ satisfying $0 \in G_n$ for all $n \in \mathbb{N}$. Consider

$$\Omega := \{ z \in \mathbb{C} \mid \exists r > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N : \overline{D(z,r)} \subset G_n \}$$

$$= \bigcup_{N \in \mathbb{N}} \operatorname{int} \Big(\bigcap_{n \geq N} G_n \Big),$$

which is an open set. We say that

(i) $(G_n)_{n=1}^{\infty}$ has a kernel (w.r.t. 0), if $0 \in \Omega$.

In this case, the connected component G of Ω that contains 0 is called the kernel of $(G_n)_{n=1}^{\infty}$. We write then

$$G = \ker (G_n)_{n=1}^{\infty}.$$

If $(G_{n_k})_{k=1}^{\infty}$ is any subsequence of $(G_n)_{n=1}^{\infty}$, then it also has a kernel and it holds that

$$\ker (G_n)_{n=1}^{\infty} \subseteq \ker (G_{n_k})_{k=1}^{\infty} \qquad (1)$$

(ii) $(G_n)_{n=1}^{\infty}$ is convergent to its kernel, if $(G_n)_{n=1}^{\infty}$ has a kernel $G = \ker (G_n)_{n=1}^{\infty}$ and equality holds in (1) for each subsequence $(G_{n_k})_{k=1}^{\infty}$. We write then

$$G_n \xrightarrow{n \to \infty} G.$$

3.2 Examples:

(i) If $(G_n)_{n=1}^{\infty}$ is any sequence of domains with

$$0 \in G_1 \subseteq G_2 \subseteq G_3 \subseteq \ldots \subseteq G_n \subseteq G_{n+1} \subseteq \ldots,$$

then

$$G_n \xrightarrow{n \to \infty} G = \bigcup_{N=1}^{\infty} G_N.$$

(ii) Put

$$Q := \{ z \in \mathbb{C} \mid -1 < \mathrm{Re}(z) < 1, -1 < \mathrm{Im}(z) < 1 \}$$

and define for $n \in \mathbb{N}$

$$G_n := \begin{cases} \mathbb{D}, & \text{if } n \text{ is odd} \\ Q, & \text{if } n \text{ is even} \end{cases}.$$

Then $\ker (G_n)_{n=1}^{\infty} = \mathbb{D}$, but

$$\ker (G_{2k-1})_{k=1}^{\infty} = \mathbb{D} \subsetneq Q = \ker (G_{2k})_{k=1}^{\infty}.$$

3.3. Lemma

Let $(f_n)_{n=1}^{\infty}$ be a sequence of schlicht holomorphic functions $f_n : \mathbb{D} \to \mathbb{C}$ that converges compactly (i.e., uniformly on compact subsets of $\mathbb{D}$) to some schlicht holomorphic function $f : \mathbb{D} \to \mathbb{C}$. Then, for any given compact set $K \subset f(\mathbb{D})$, there is some $N \in \mathbb{N}$, such that

$$K \subset f_n(\mathbb{D}) \qquad \text{for all } n \geq N.$$

Proof: Exercise! $\square$

3.4. Theorem (Carathéodory, 1912)

Let $(G_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ be given like in (*). Then the following statements hold true:

(i) $(f_n)_{n=1}^\infty$ converges compactly to a schlicht holomorphic function $f: \mathbb{D} \to \mathbb{C}$, if and only if $(G_n)_{n=1}^\infty$ has a kernel $G = \ker (G_n)_{n=1}^\infty \subsetneq \mathbb{C}$ and is convergent to its kernel.

In this case, $f(\mathbb{D}) = G$ holds, i.e., f is the unique biholomorphic mapping $f: \mathbb{D} \to G$ satisfying $f(0) = 0$ and $f'(0) > 0$.

(ii) $(f_n)_{n=1}^\infty$ converges compactly to 0, if and only if no subsequence of $(G_n)_{n=1}^\infty$ has a kernel.

3.5 Remark:

There is a notion of compact convergence for sequences $(g_n)_{n=1}^\infty$ of holomorphic functions $g_n : G_n \to \mathbb{C}$, such that in the situation (i) even

$$f_n^{-1} \xrightarrow[\text{compact}]{} f^{-1} \quad \text{as} \quad n \to \infty.$$

Proof of Theorem 3.4:

We only prove (i) and leave (ii) as an exercise.

"$\Longrightarrow$": Suppose that $(f_n)_{n=1}^\infty$ converges compactly to some schlicht function $f: \mathbb{D} \to \mathbb{C}$.

① <u>Claim</u>: $(G_n)_{n=1}^\infty$ has a kernel $G = \ker(G_n)_{n=1}^\infty$ and it holds true that $f(\mathbb{D}) \subseteq G$.

<u>Proof</u>: Take $w \in f(\mathbb{D})$. Since $f(\mathbb{D})$ is open, there is some $r > 0$, such that

$$\overline{D(w,r)} \subset f(\mathbb{D}).$$

By Lemma 3.3, we find $N \in \mathbb{N}$, such that

$$\overline{D(w,r)} \subset f_n(\mathbb{D}) = G_n \qquad \forall n \geq N.$$

Thus

$$w \in \Omega := \bigcup_{N \in \mathbb{N}} \operatorname{int}\Big(\bigcap_{n \geq N} G_n\Big).$$

Hence $0 \in f(\mathbb{D}) \subseteq \Omega$, so that $(G_n)_{n=1}^\infty$ has a kernel $G = \ker(G_n)_{n=1}^\infty$, and since $f(\mathbb{D})$ is connected, it follows $f(\mathbb{D}) \subseteq G$. □

② <u>Claim</u>: We have that $f(\mathbb{D}) = G$.

<u>Proof</u>: Take $w \in G$. By definition of the kernel, we find a bounded domain

$$D \subset \overline{D} \subset G$$

with $0 \in D$ and $w \in D$, and by Lemma 3.3 some $N \in \mathbb{N}$, so that

$$\overline{D} \subset G_n \qquad \forall n \geq N.$$

Consider the sequence $(g_n)_{n=N}^\infty$ given by

$$g_n := f_n^{-1}\big|_D : D \to \mathbb{D}.$$

Then $(g_n)_{n=N}^{\infty}$ is (locally) bounded, hence by Montel's theorem, has a subsequence $(g_{n_k})_{k=1}^{\infty}$ that converges compactly to some function $g: D \to \overline{\mathbb{D}}$. Since

$$g(0) = \lim_{k\to\infty} g_{n_k}(0) = 0,$$

the open mapping theorem (Theorem 1.2) yields that in fact $g: D \to \mathbb{D}$. Now, since with $(f_n)_{n=1}^{\infty}$ also $(f_{n_k})_{k=1}^{\infty}$ converges compactly to f, we infer from $g_{n_k}(w) \to g(w) \in \mathbb{D}$ that

$$w = f_{n_k}(g_{n_k}(w)) \longrightarrow f(g(w))$$

as $k \to \infty$; thus

$$w = f(g(w)) \in f(\mathbb{D}).$$

Hence, $G \subseteq f(\mathbb{D})$ and ① yields $f(\mathbb{D}) = G$.

$\square$

③ Since the arguments of ① and ② can be applied to any subsequence $(f_{n_k})_{k=1}^{\infty}$, we get

$$f(\mathbb{D}) = \ker (G_{n_k})_{k=1}^{\infty} = G,$$

i.e. $G_n \to G$ as $n \to \infty$, with $G = f(\mathbb{D}) \subsetneq \mathbb{C}$ and $f(0) = 0$, $f'(0) > 0$.