2.12 Remark:

(i) By careful examination of the proofs, one sees that equality in one of the four inequalities of Theorem 2.10 and 2.11 at some \( z \in D \setminus \mathcal{E}_0 \) enforces that \( f \) is a rotation of the Kolos function.

(ii) Using Theorem 2.11 and Hurwitz's theorem, one sees that \( f \) is in a compact subset of the complete metric space \( (\mathcal{O}(O), d) \), where \( d \) denotes the metric of compact convergence.

(iii) The coefficient functionals

\[ a_n : (\mathcal{O}(O), d) \to \mathbb{C}, \ f \mapsto a_n(f) \]

are continuous, since

\[ a_n(f) = \frac{1}{2\pi i} \int_{\partial D(0, r)} \frac{f(z)}{z^{n+1}} \, dz \]

\[ \Rightarrow \ |a_n(f)| \leq \frac{1}{r^n} \|f\|_{\mathcal{O}(O)}, \ r \in (0, 1). \]

Thus, \( b_3(ii) : \sup_{f \in \mathcal{O}} |a_n(f)| < \infty \ \forall n \geq 2. \)

(iv) In fact, one can show that (Littlewood, 1925)

\[ |a_n| < c_n \quad \forall n \geq 2 \]

for all \( f \in \mathcal{O} \). Thus, the Bieberbach conjecture gives the correct order \( |a_n| = O(n) \).
(v) Even better (Hayman, 1955) for each \( f \in \mathcal{G} \):

\[
\lambda := \lim_{n \to \infty} \frac{|a_n|}{n}
\]

exists and \( \lambda \leq 1 \), with equality if and only if \( f \) is a rotation of the Koebe function.

The **Hayman index** \( \lambda \) can be computed by

\[
\lim_{r \to 1^+} \frac{1-r^2}{r} \max_{\theta \in [0,2\pi]} |f(re^{i\theta})| = \lambda.
\]

(vi) Another interesting application of Prop. 2.9 is the following:

For all \( 0 < r \leq r_0 := 2 - \sqrt{3} \), the disc \( D(0, r) \) is mapped by each \( f \in \mathcal{G} \) to a convex domain; this fails for \( r > r_0 \).
3. The Carathéodory kernel theorem

Let us consider the following situation:

- Let \((G_n)_{n=1}^{\infty}\) be a sequence of simply connected domains \(G_n \subset \mathbb{C}\) with \(0 \in G_n\).
- Let \((f_n)_{n=1}^{\infty}\) be the corresponding sequence of biholomorphic mappings \(f_n : D \to G_n\) satisfying \(f_n(0) = 0\) and \(f_n'(0) > 0\) (see Theorem 1.5).

Our goal is to introduce a suitable notion of convergence for \((G_n)_{n=1}^{\infty}\), such that

\[
\begin{align*}
&\lim_{n \to \infty} f_n \to f \text{ compactly on } D \\
\iff &\lim_{n \to \infty} G_n \to G \text{ with } G = f(D) 
\end{align*}
\]

3.1. Definition:

Let \((G_n)_{n=1}^{\infty}\) be any sequence of domains in \(\mathbb{C}\) satisfying \(0 \in G_n\) for all \(n \in \mathbb{N}\). Consider

\[
\Omega := \left\{ z \in \mathbb{C} \mid \exists r > 0 \ \forall N \in \mathbb{N} \ \forall n \geq N : \overline{D(z, r)} \subset G_n \right\}
\]

\[
= \bigcup_{N \in \mathbb{N}} \text{int} \left( \bigcap_{n \geq N} G_n \right),
\]

which is an open set. We say that
(i) \((G_n)_{n=1}^{\infty}\) has a kernel (w.r.t. 0), if \(0 \in \Omega\). \(\exists \ \Omega\)

In this case, the connected component \(\overline{G}_0\) of \(\Omega\) that contains 0 is called the kernel of \((G_n)_{n=1}^{\infty}\). We write then

\[ G_0 = \ker (G_n)_{n=1}^{\infty}. \]

If \((G_{n_k})_{k=1}^{\infty}\) is any subsequence of \((G_n)_{n=1}^{\infty}\), then it also has a kernel and it holds that

\[ \ker (G_n)_{n=1}^{\infty} \subseteq \ker (G_{n_k})_{k=1}^{\infty}. \] (1)

(ii) \((G_n)_{n=1}^{\infty}\) is convergent to its kernel, if \((G_n)_{n=1}^{\infty}\) has a kernel \(G = \ker (G_n)_{n=1}^{\infty}\) and equality holds in (1) for each subsequence \((G_{n_k})_{k=1}^{\infty}\). We write then

\[ G_n \xrightarrow{n \to \infty} G_0. \]

3.2 Examples:

(i) If \((G_n)_{n=1}^{\infty}\) is any sequence of domains with

\[ 0 \in G_1 \subseteq G_2 \subseteq G_3 \subseteq \ldots \subseteq G_n \subseteq G_{n+1} \subseteq \ldots, \]

then

\[ G_n \xrightarrow{n \to \infty} G_0 = \bigcup_{N=1}^{\infty} G_N. \]
(ii) Put
\[ Q := \{ z \in \mathbb{C} \mid -1 < \text{Re}(z) < 1, -1 < \text{Im}(z) < 1 \} \]
and define for \( n \in \mathbb{N} \)
\[ G_n := \begin{cases} D, & \text{if } n \text{ is odd} \\ Q, & \text{if } n \text{ is even} \end{cases} \]
Then \( \ker (G_n)_{n=1}^\infty = D \), but
\[ \ker (G_{2n-1})_{n=1}^\infty = D \supsetneq Q = \ker (G_{2n})_{n=1}^\infty. \]

3.3. Lemma

Let \( (f_n)_{n=1}^\infty \) be a sequence of holomorphic functions \( f_n : D \to \mathbb{C} \) that converges compactly (i.e., uniformly on compact subsets of \( D \)) to some holomorphic function \( f : D \to \mathbb{C} \). Then, for any given compact set \( K \subset f(D) \), there is some \( N \in \mathbb{N} \) such that
\[ K \subset f_n(D) \quad \text{for all } n \geq N. \]

Proof: Exercise! \( \square \)

3.4. Theorem (Carathéodory, 1912)

Let \( (G_n)_{n=1}^\infty \) and \( (f_n)_{n=1}^\infty \) be given like in (*)

Then the following statements hold true:
(i) \((f_n)_{n=1}^\infty\) converges compactly to a sublinear holomorphic function \(f : \overline{D} \to C\), if and only if \((G_n)_{n=1}^\infty\) has a kernel \(G = \ker (G_n)_{n=1}^\infty \neq C\) and is convergent to its kernel.

In this case, \(f(\overline{D}) = G\) holds, i.e., \(f\) is the unique biholomorphic mapping \(f : \overline{D} \to G\) satisfying \(f(0) = 0\) and \(f'(0) > 0\).

(ii) \((f_n)_{n=1}^\infty\) converges compactly to 0, if and only if no subsequence of \((G_n)_{n=1}^\infty\) has a kernel.

3.5 Remark:
There is a notion of compact convergence for sequences \((g_n)_{n=1}^\infty\) of holomorphic functions \(g_n : G_n \to C\), such that in the situation (i) even
\[ f_n^{-1} \to f^{-1} \quad \text{as} \quad n \to \infty. \]

Proof of Theorem 3.4:
We only prove (i) and leave (ii) as an exercise.

\(\Rightarrow\): Suppose that \((f_n)_{n=1}^\infty\) converges compactly to some sublinear function \(f : \overline{D} \to C\).
(1) **Claim:** \((G_n)_{n=1}^{\infty}\) has a kernel \(G = \ker (G_n)_{n=2}^{\infty}\), and it holds true that \(f(N) \subseteq G\).

**Proof:** Take \(w \in f(N)\). Since \(f(N)\) is open, there is some \(r > 0\), such that \(D(w, r) \subseteq f(N)\).

By Lemma 3.3, we find \(N \in N\), such that \(D(w, r) \subseteq f_n(N) = G_n, \quad \forall n \geq N\).

Thus, \(w \in \Omega := \bigcup_{N \in N} \text{int} \left( \bigcap_{n \geq N} G_n \right)\).

Hence, \(0 \in f(N) \subseteq \Omega\), so that \((G_n)_{n=1}^{\infty}\) has a kernel \(G = \ker (G_n)_{n=2}^{\infty}\), and since \(f(N)\) is connected, it follows \(f(N) \subseteq G\).

(2) **Claim:** We have that \(f(N) = G\).

**Proof:** Take \(w \in G\). By definition of the kernel, we find a bounded domain \(D \subseteq \overline{D} \subseteq G\) with \(0 \in D\) and \(w \in D\), and by Lemma 3.3 some \(N \in N\), so that \(\overline{D} \subseteq G_n, \quad \forall n \geq N\).

Consider the sequence \((g_n)_{n=1}^{\infty}\) given by \(g_n := f_n^{(-1)}|_D : D \to D\).
Then \((g_n)_{n=1}^\infty\) is (locally) bounded, hence \[3-6\] by Montel's theorem, has a subsequence \((g_{n_k})_{k=1}^\infty\) that converges compactly to some function \(g : D \to \overline{D}\). Since
\[
g(0) = \lim_{k \to \infty} g_{n_k}(0) = 0,
\]
the open mapping theorem (Theorem 1.2) yields that in fact \(g : D \to D\). Now, since with \((f_n)_{n=1}^\infty\), also \((f_{n_k})_{k=1}^\infty\), converges compactly to \(f\), we infer from \(g_{n_k}(w) \to g(w) \in \overline{D}\) that
\[
w = f_{n_k}(g_{n_k}(w)) \to f(g(w))
\]
as \(k \to \infty\); thus
\[
w = f(g(w)) \in f(\overline{D}).
\]
Hence, \(G \subseteq f(\overline{D})\) and \(\circ\) yields \(f(\overline{D}) = G\).

\(3\) Since the arguments of \(1\) and \(2\) can be applied to any subsequence \((f_{n_k})_{k=1}^\infty\), we get
\[
f(\overline{D}) = \ker (G_{n_k})_{k=1}^\infty = G,
\]
\(i.e.\) \(G_n \to G\) as \(n \to \infty\), with \(G = f(\overline{D}) \subseteq C\) and \(f(0) = 0\), \(f'(0) > 0\).