"$\Leftarrow$": Suppose now that

$$G_n \xrightarrow{n\to\infty} G = \ker(G_n)_{n=1}^{\infty} \quad \text{with} \quad G \subsetneq \mathbb{C}.$$

④ <u>Claim</u>: $(f_n)_{n=1}^{\infty}$ is locally bounded (and hence, by Montel's theorem, a normal family).

<u>Proof</u>: Since $\frac{1}{f_n'(0)} f_n \in \mathcal{S}$, Theorem 2.8 says

$$D(0, \tfrac{1}{4} f_n'(0)) \subseteq G_n$$

Thus,

$$M = \sup_{n\in\mathbb{N}} f_n'(0) < \infty,$$

since otherwise, we could find a subsequence $(f_{n_k})_{k=1}^{\infty}$ with $f_{n_k}'(0) \to \infty$ so that

$$\ker(G_{n_k})_{k=1}^{\infty} = \mathbb{C},$$

in contradiction to $G_n \xrightarrow{n\to\infty} G \neq \mathbb{C}$.

Now, Theorem 2.11 gives us

$$|f_n(z)| \le f_n'(0) \frac{|z|}{(1-|z|)^2} \le M \frac{|z|}{(1-|z|)^2}$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$. Thus, $(f_n)_{n=1}^{\infty}$ is locally bounded.

$\square$

⑤ <u>Claim</u>: $(f_n)_{n=1}^{\infty}$ converges compactly.

<u>Proof</u>: In view of ④, it suffices to prove that if $(f_{n_1(k)})_{k=1}^{\infty}$ and $(f_{n_2(k)})_{k=1}^{\infty}$ are convergent

subsequences of $(f_n)_{n=1}^{\infty}$, say

$$f_{n_1(k)} \to h_1 \quad \text{and} \quad f_{n_2(k)} \to h_2,$$

then necessarily $h_1 = h_2$.
(Montel's convergence criterion)

In ②, we have seen that

$$h_1(\mathbb{D}) = \ker (G_{n_1(k)})_{k=1}^{\infty},$$

$$h_2(\mathbb{D}) = \ker (G_{n_2(k)})_{k=1}^{\infty},$$

so that $G_n \to G$ enforces

$$h_1(\mathbb{D}) = G = h_2(\mathbb{D}).$$

Since furthermore

$$h_1(0) = 0 \quad \text{and} \quad h_1'(0) > 0,$$
$$h_2(0) = 0 \quad \text{and} \quad h_2'(0) > 0,$$

the uniqueness statement in the Riemann mapping theorem (Theorem 1.5) gives $h_1 = h_2$. $\square$

This concludes the proof of Theorem 3.4. $\square$

3.6. Remark:

A Jordan arc means an injective continuous map

$$\gamma : [0, T) \to \mathbb{C} \quad \text{with} \quad \lim_{t \nearrow T} |\gamma(t)| = \infty$$

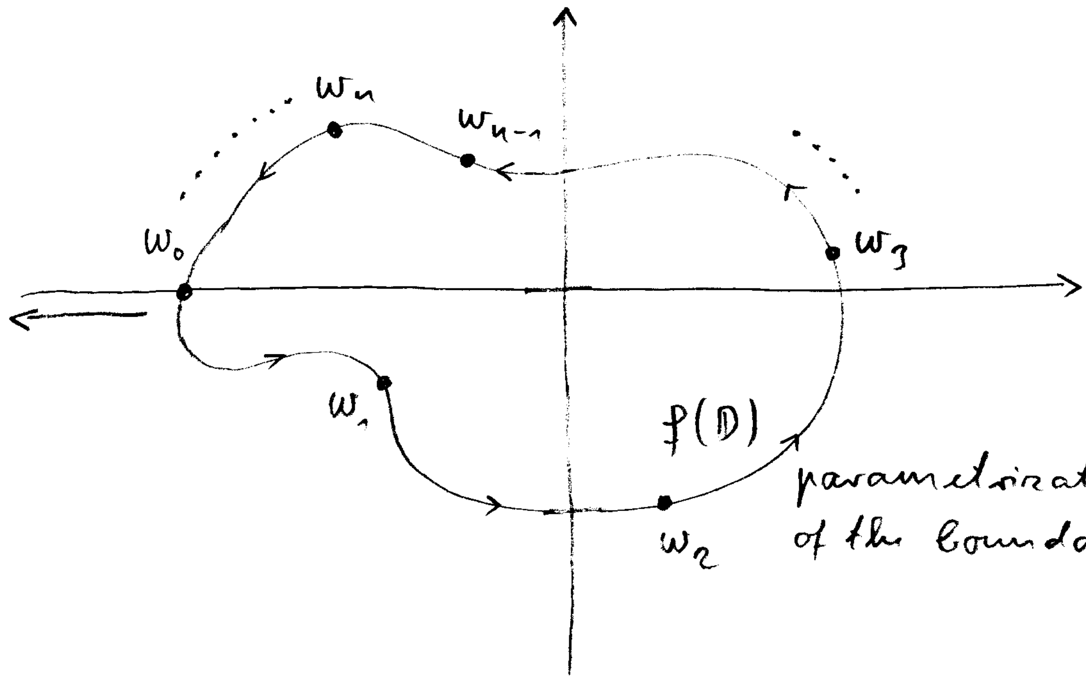
for some $0 < T \leq \infty$; in such cases, $G = \mathbb{C} \setminus \gamma([0,T))$

is called slit domain. Accordingly, $f \in \mathcal{S}$ is called slit mapping, if $G = f(\mathbb{D})$ is a slit domain.

With the help of Theorem 3.4., one can show that for each $f \in \mathcal{S}$, there exists a sequence $(f_n)_{n=1}^{\infty}$ of slit mappings in $\mathcal{S}$ that converges compactly to f.

Slit mappings are thus "generic" functions in $\mathcal{S}$.

For $f \in \mathcal{S}$ with $f(\mathbb{D})$ being a Jordan domain with smooth boundary, we may construct slit domains G_n as follows:



γ_n : from w_n along the boundary back to w_0; then from w_0 to $-\infty$ along the real axis.

$G_n := \mathbb{C} \setminus \gamma_n([0,\infty)) \implies \ker (G_n)_{n=1}^{\infty} = f(\mathbb{D})$.

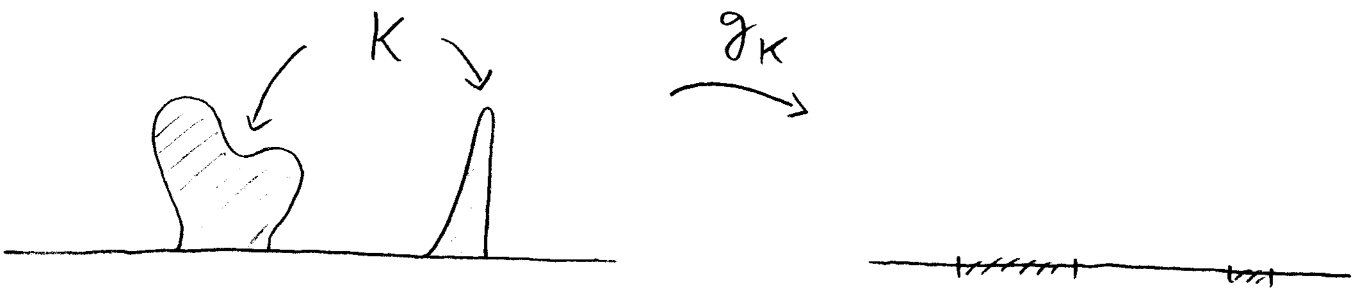
4. Mapping-out functions and half-plane capacity

Let $\mathbb{H}$ denote the complex upper half-plane, i.e.

$$\mathbb{H} := \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}.$$

4.1. Definition:

A subset $K \subset \mathbb{H}$ is called compact $\mathbb{H}$-hull, if K is bounded and $\mathbb{H} \setminus K$ is a simply connected domain.



4.2 Theorem:

For each compact $\mathbb{H}$-hull K, there is a unique biholomorphic mapping

$$g_K : \mathbb{H} \setminus K \to \mathbb{H}$$

that satisfies the hydrodynamic normalization

$$\lim_{|z| \to \infty} (g_K(z) - z) = 0.$$

We call g_K the mapping-out function of K.

Moreover, the limit

$$\mathrm{hcap}(K) := \lim_{|z|\to\infty} z\,(g_K(z) - z)$$

exists and is a real number. We call $\mathrm{hcap}(K)$ the <u>half-plane capacity of K</u>.

The proof requires some preparations.

<u>4. 3. Notation:</u>

If $G \subsetneqq \mathbb{H}$ is a simply connected domain, then we put $\widetilde{G} := G \cup G^\circ \cup G^*$, where

$$G^* := \{\bar{z} \mid z \in G\} \qquad \text{and}$$

$$G^\circ := \{x \in \mathbb{R} \mid \exists\, \varepsilon > 0 : D(x,\varepsilon) \cap \mathbb{H} \subset G\}.$$

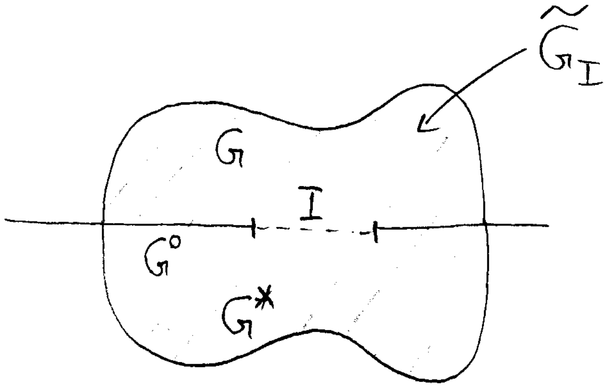
If $I \subseteq G^\circ$ is an open interval, we put

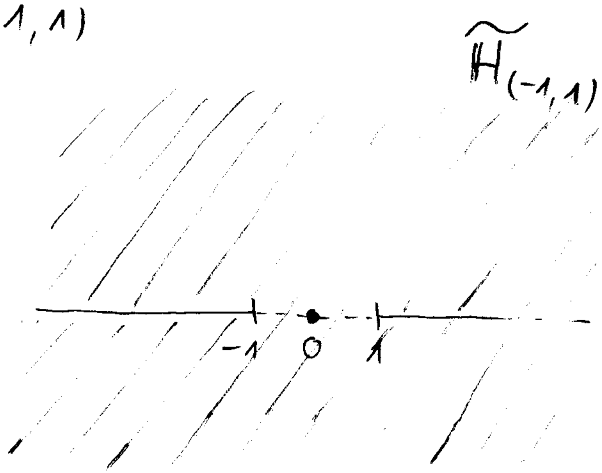
$$\widetilde{G}_I := G \cup I \cup G^*.$$

<u>4. 4. Proposition:</u>

Let $G \subseteq \mathbb{H}$ be a simply connected domain and let $I \subsetneqq \mathbb{R}$ be an open interval with $I \subseteq G^\circ$. Then, for any $x \in I$, there exists a unique biholomorphic mapping $\phi : G \to \mathbb{H}$ that extends to a homeomorphism $G \cup I \to \mathbb{H} \cup (-1,1)$ taking x to 0. Moreover, ϕ extends by reflection to a biholomorphic mapping

$$\tilde{\phi} : \tilde{G}_I \longrightarrow \tilde{\mathbb{H}}_{(-1,1)}$$





<u>Proof</u>: Exercise.

<u>Proof of Theorem 4.2:</u>

Consider the simply connected domain

$$G := \left\{ -\frac{1}{z} \;\middle|\; z \in \mathbb{H} \setminus K \right\}$$

Since $0 \in G^\circ$, we may choose an open interval I with the property that $0 \in I \subseteq G^\circ$. Let

$$\phi : G \to \mathbb{H}$$

be the biholomorphic mapping from Proposition 4.4 and let

$$\tilde{\phi} : \tilde{G}_I \longrightarrow \tilde{\mathbb{H}}_{(-1,1)}, \quad \tilde{\phi}(0) = 0$$

be its extension obtained by reflection. On any $D(0,\varepsilon) \subset \tilde{G}_I$,

$$\tilde{\phi}(z) = b_1 z + b_2 z^2 + b_3 z^3 + O(|z|^4)$$

where $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, as $\tilde{\phi}$ is reflection symmetric, i.e.

$$\overline{\tilde{\phi}(z)} = \tilde{\phi}(\bar{z}) \quad \text{for all } z \in \tilde{G}_I .$$

Now, consider $\tilde{\psi} \in \mathcal{O}(D(0,\varepsilon))$ given by

$$\tilde{\psi}(z) = \frac{\tilde{\phi}(z)}{z} = b_1 + b_2 z + b_3 z^2 + O(|z|^3)$$

Then (note that $b_1 = \tilde{\phi}'(0) \neq 0$)

$$\frac{z}{\tilde{\phi}(z)} = \frac{1}{\tilde{\psi}(z)} = \frac{1}{b_1} - \frac{b_2}{b_1^2} z + \left(\frac{b_3}{b_1^2} - \frac{b_2^2}{b_1^3} \right) z^2 + O(|z|^3),$$

so that

$$\frac{b_1}{\tilde{\phi}(z)} = \frac{1}{z} - \frac{b_2}{b_1} + \left(\frac{b_3}{b_1} - \left(\frac{b_2}{b_1} \right)^2 \right) z + O(|z|^2)$$

Thus, for $|z| > \frac{1}{\varepsilon}$, we get

$$-\frac{b_1}{\tilde{\phi}(-\frac{1}{z})} = z + \frac{b_2}{b_1} - \left(\frac{b_3}{b_1} - \left(\frac{b_2}{b_1} \right)^2 \right) \frac{1}{z} + O(|z|^{-2}).$$

Hence,

$$g_K : \mathbb{H} \setminus K \longrightarrow \mathbb{H}, \quad z \longmapsto -\frac{b_1}{\tilde{\phi}(-\frac{1}{z})} - \frac{b_2}{b_1}$$

is a well-defined biholomorphic mapping and satisfies

$$g_K(z) - z \longrightarrow 0 \quad \text{as} \quad |z| \to \infty$$

and

$$z\left(g_K(z) - z\right) \longrightarrow \left(\frac{b_2}{b_1}\right)^2 - \frac{b_3}{b_1} \in \mathbb{R} \quad \text{as } |z| \to \infty.$$

Uniqueness is left as an exercise. $\square$