1.11. Remark

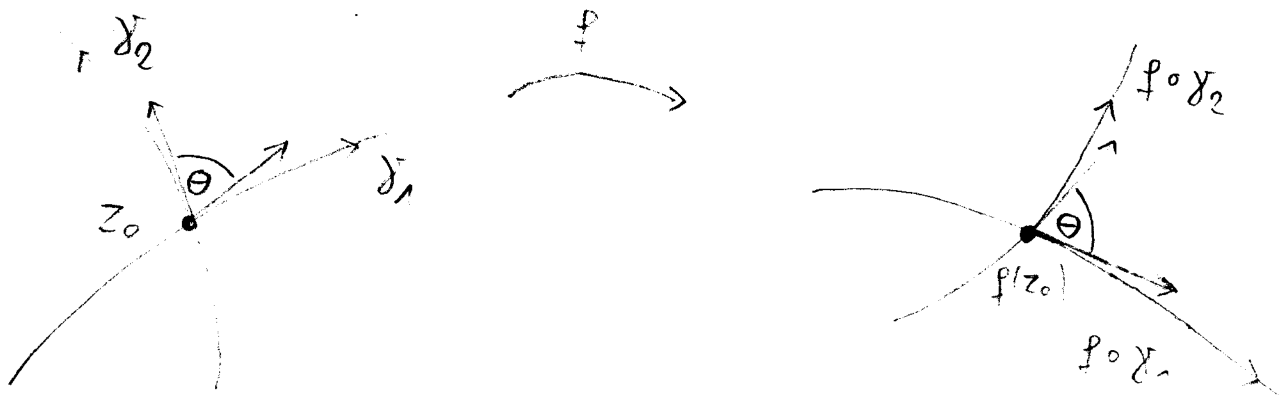
Suppose that $f : \Omega \to \mathbb{C}$ is locally conformal. Consider smooth paths $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \Omega$ with $z_0 := \gamma_1(0) = \gamma_2(0)$ and

$$\gamma_1'(0) \neq 0 \qquad \text{and} \qquad \gamma_2'(0) \neq 0 .$$

Then, for $j = 1, 2$,

$$\begin{aligned}(f \circ \gamma_j)'(0) &= f'(\gamma_j(0))\, \gamma_j'(0) \\ &= f'(z_0)\, \gamma_j'(0) = Df(z_0)\, \gamma_j'(0),\end{aligned}$$

hence $\Theta := \measuredangle\big((f\circ\gamma_1)'(0), (f\circ\gamma_2)'(0)\big) = \measuredangle\big(\gamma_1'(0), \gamma_2'(0)\big)$.



2. Elementary growth and distortion theorems

This chapter is about the class (see Def. 0.1 (i))

$$\mathcal{S} = \{ f \in \mathcal{O}(\mathbb{D}) \text{ schlicht} \mid f(0) = 0, f'(0) = 1 \}.$$

In order to derive some of its fundamental properties, it is appropriate to study first another class of functions.

2.1. Definition:

Put $\Delta := \mathbb{C} \setminus \overline{\mathbb{D}} = \{ z \in \mathbb{C} \mid |z| > 1 \}$. We denote by

(i) $\Sigma \subset \mathcal{O}(\Delta)$ the set of all schlicht functions

$$g : \Delta \to \mathbb{C}$$

with a simple pole at ∞ that satisfy

$$\lim_{\zeta \to \infty} \frac{g(\zeta)}{\zeta} = 1.$$

(ii) $\Sigma' \subset \mathcal{O}(\Delta)$ the set of all functions of the form

$$g : \Delta \to \mathbb{C}, \ \zeta \mapsto \frac{1}{f\left(\frac{1}{\zeta}\right)} \qquad (1)$$

for some $f \in \mathcal{S}$.

Obviously, we have that $\Sigma' \subsetneq \Sigma$.

2.2. Remark:

Each $g \in \Sigma$ has a Laurent expansion of the form

$$g(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}. \qquad (2)$$

If $g \in \Sigma'$ is associated to $f \in \mathcal{S}$ via (1), then

$$a_{n+1} = - \sum_{k=1}^{n} b_{n-k} a_k \qquad \forall n \in \mathbb{N},$$

where we suppose that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \ a_1 := 1.$$

2.3. Theorem (Grönwall's area theorem, 1914)

Let $g \in \Sigma$ be given and consider its Laurent expansion (2). Then

$$\sum_{n=1}^{\infty} n |b_n|^2 \leq 1.$$

The proof requires some preparation.

2.4. Theorem (Jordan curve theorem)

Let Γ be a Jordan curve, i.e., the image $\Gamma = \tilde{\gamma}(\mathbb{T})$ of an injective continuous map $\tilde{\gamma} : \mathbb{T} \to \mathbb{C}$ on $\mathbb{T} := \partial\mathbb{D} = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Then $\mathbb{C} \setminus \Gamma$ consists of exactly two connected components $G_0, G_1 \subset \mathbb{C}$. One of theses components, say G_0, is bounded (called the <u>interior of Γ</u>), and the other one, G_1, is unbounded (called the <u>exterior of Γ</u>). Moreover, it holds true that $\partial G_0 = \Gamma = \partial G_1$.

Note that the Jordan curve Γ is the image of the closed path (without self-intersection points)

$$\gamma : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto \tilde{\gamma}(e^{it}) .$$

Now, if we suppose that γ is even a piecewise smooth path, then <u>Green's theorem</u> yields that for each open set $\Omega \subseteq \mathbb{C}$ with $\overline{G_0} \subset \Omega$ and for each $f \in C^1(\Omega)$

$$\pm \int_\gamma f(z)\, dz = 2i \int_{G_0} \frac{\partial f}{\partial \bar{z}}(z)\, d\lambda^2(z)$$

sign depending on the orientation of γ

Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$

In particular, for $f : \mathbb{C} \to \mathbb{C}, z \mapsto \bar{z}$, we get that

$$\pm \lambda^2(G_0) = \frac{1}{2i} \int_\gamma \bar{z}\, dz \tag{3}$$

$$= \frac{1}{2i} \int_0^{2\pi} \overline{\gamma(t)}\, \gamma'(t)\, dt =: A(\gamma) .$$

Proof of Theorem 2.3:

Choose any $r > 1$. Since $g : \Delta \to \mathbb{C}$ is schlicht,

$$\gamma_r : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto g(re^{it})$$

gives a smooth parametrization of a Jordan curve Γ_r; denote by G_r the interior of Γ.

We write g like in (2). We have then

$$\gamma_r'(t) = g'(re^{it})\, i r e^{it}$$

$$= i\left(re^{it} - \sum_{m=1}^{\infty} m\, b_m \frac{1}{r^m} e^{-imt}\right)$$

and

$$\overline{\gamma_r(t)} = re^{-it} + \sum_{n=0}^{\infty} \overline{b}_n \left(re^{-it}\right)^{-n}$$

$$= re^{-it} + \sum_{n=0}^{\infty} \overline{b}_n \frac{1}{r^n} e^{int},$$

so that

$$\frac{1}{i}\, \overline{\gamma_r(t)}\, \gamma_r'(t) = r^2 + \sum_{n=0}^{\infty} \overline{b}_n \frac{1}{r^{n-1}} e^{i(n+1)t}$$

$$- \sum_{m=1}^{\infty} m\, b_m \frac{1}{r^{m-1}} e^{-i(m+1)t}$$

$$- \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m\, \overline{b}_n b_m \frac{1}{r^{n+m}} e^{i(n-m)t}.$$

Using that $\frac{1}{2\pi}\int_0^{2\pi} e^{ikt}\,dt = \delta_{k,0}$ for all $k \in \mathbb{Z}$, we get

$$A(\gamma_r) = \frac{1}{2i} \int_0^{2\pi} \overline{\gamma_r(t)}\, \gamma_r'(t)\, dt$$

$$= \pi \left(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 \frac{1}{r^{2n}} \right) \qquad (4)$$

Now, we see that $A(\gamma_r) \to \infty$ as $r \to \infty$.
Since by (3), $A(\gamma_r) = \pm \underbrace{\lambda^2(G_r)}_{>0}$, it follows that $A(\gamma_r) > 0$ for all $r > 1$. Thus, by (4):

$$\sum_{n=1}^{\infty} n |b_n|^2 \frac{1}{r^{2(n+1)}} < 1 \qquad \forall r > 1.$$

By the monotone convergence theorem, we get

$$\sum_{n=1}^{\infty} n |b_n|^2 \leq 1,$$

as desired. $\square$

2.5. Corollary:

Let $g \in \Sigma$ be given and consider its Laurent expansion (2). Then

$$|b_1| \leq 1,$$

with equality if and only if g is of the form

$$g(\zeta) = \zeta + b_0 + \frac{b_1}{\zeta} \quad \text{with } b_0, b_1 \in \mathbb{C},\ |b_1| = 1. \qquad (5)$$

In this case, g maps Δ onto the complement of some line of length 4.

<u>Proof</u>:

By Theorem 2.3, we have that

$$|b_1|^2 \leq \sum_{n=1}^{\infty} n|b_n|^2 \leq 1, \qquad (6)$$

hence $|b_1| \leq 1$. Equality holds if and only if g is of the form (5), since (6) enforces

$$b_n = 0 \qquad \forall n \geq 2$$

if $|b_1| = 1$. The additional statement follows, since

$$g_0 : \Delta \to \mathbb{C} \setminus [-2,2], \; \zeta \mapsto \zeta + \frac{1}{\zeta}$$

is bijective and

$$g(\zeta) = b_0 + e^{i\frac{\theta}{2}} g_0(e^{-i\frac{\theta}{2}}\zeta), \quad \zeta \in \Delta$$

holds with $b_1 = e^{i\theta}$, $\theta \in [0, 2\pi)$.

$\square$

Now, we may return to the class $\mathcal{S}$.

<u>2.6. Theorem</u> (Bieberbach, 1916):

Let $f \in \mathcal{S}$ be given and consider its power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Then

$$|a_2| \leq 2,$$

with equality if and only if f is a rotation of the Koebe function k_0, i.e. $f = k_\Theta$ for some $\Theta \in [0, 2\pi)$

Recall from Problem 2, Assignment 1B, that

$$k_\Theta : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{(1 - z e^{i\Theta})^2} \quad \text{and}$$

$$k_0 : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{(1-z)^2}.$$