COMPLEX ANALYSIS II

Introduction to geometric function theory and Schramm-Loewner evolution

0. Introduction

function theory $\iff$ geometric function theory

theory of holomorphic functions $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$ open, ...

$\Theta(\Omega)$ $\iff$ $\Theta(\Omega_1, \Omega_2)$

typical questions: curve integrals, antiderivatives, series expansion, residues, ...

typical questions: injectivity, surjectivity, bijectivity, $f(\Omega_1), f^{-1}$, ...

"Holomorphic functions are the isomorphisms of function theory"

$\exists f : \Omega_1 \to \Omega_2$ biholomorphic (i.e. $f$ bijective, $f, f^{-1}$ hol.) $\Rightarrow$ $\Omega_1$ and $\Omega_2$ have "similar" properties in the sense of complex analysis
Guiding question of geom. function theory

How are the properties of \( f \) as a holomorphic function \( f : \Omega \rightarrow \mathbb{C} \) affected by its properties as a (holomorphic) mapping \( f : \Omega \rightarrow \Omega \)?

0.1. Definition:

(i) Let \( \Omega \subseteq \mathbb{C} \) be open. A holomorphic function \( f : \Omega \rightarrow \mathbb{C} \) is called \textit{univalent}, if it is injective.

(ii) Consider \( D := \{ z \in \mathbb{C} \mid |z| < 1 \} \). We put \( \mathcal{S} := \{ f \in \mathcal{O}(D) \text{ univalent} \mid f(0) = 0, f'(0) = 1 \} \).

0.2. Theorem (Bieberbach conjecture, Ludwig Bieberbach, 1916, Louis de Branges, 1985)

Consider the power series expansion

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D,
\]

of any \( f \in \mathcal{S} \). Then

\[
|a_n| \leq n \quad \forall n \geq 2
\]

with equality for any \( n \geq 2 \) if and only if \( f \) is a rotation of the \textit{Koebe function}. 
Our goals:

- We want to understand
  - some of the main tools of the proof and
  - how they are used in modern research.

In particular: stochastic Loewner evolution (SLE)

0.3. Example (loop-erased random walk, LERW)

Consider the random walk half-plane excursion, i.e., the process \((S_n)_{n=1}^{\infty}\) on 

\[ \mathbb{Z}^2_+ := \{(j,k) \in \mathbb{Z}^2 \mid k > 0\} \subset \mathbb{C} \]

with transition probabilities given by

\[ \begin{array}{c}
(\frac{2}{a}, \frac{b-1}{4b}) & \frac{a}{4} & (\frac{2}{a}, \frac{b}{4b}) \\
(\frac{b}{4b}, \frac{b-1}{4b}) & (\frac{b}{4b}, \frac{b}{4b}) & (\frac{b-1}{4b}, \frac{b}{4b})
\end{array} \]

Erase loops in chronological order; obtain the loop-erased random walk \((\hat{S}_n)_{n=0}^{\infty}\); linear interpolation yields \((\hat{S}_t)_{t \geq 0}\); let \((\hat{S}^\delta_t)_{t \geq 0}\) be the scaled process on \(\delta \mathbb{Z}^2_+\) for \(\delta > 0\).

The scaling limit of \((\hat{S}_t)_{t \geq 0}\) (i.e., \((\hat{S}^\delta_t)_{t \geq 0}\) for \(\delta > 0\))
is given by the chordal $\text{SLE}_2$.

(G. Lawler, O. Schramm, W. Werner, 2004)

For any $\kappa > 0$, $\text{SLE}_\kappa$ means the random path $y$ in

$H \coloneqq \{ z \in \mathbb{C} \mid \Im(z) > 0 \}$ associated to

the random collection $(g_t)_{t \geq 0}$ of biholomorphic maps $g_t : H \setminus y([0,t]) \to H$ obtained from

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \overline{y}(t)} B_t$$

$g_0(z) = z$

Note that $y : [0, \infty) \to H$ satisfies $y(0) = 0$ and is injective.

$y(t)$

$\overbrace{y(t)}^\text{one-dimensional Brownian motion}$

$\Gamma$

$g_t(y(t)) = \sqrt{\kappa} B_t$
1. Conformal maps

1.1. Reminder:

Let \( \Omega \subseteq \mathbb{C} \) be open. For \( f: \Omega \to \mathbb{C} \) and each \( z_0 \in \Omega \), the following statements are equivalent:

(i) \( f \) is complex differentiable at \( z_0 \), i.e. the limit

\[
    f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]

exists.

(ii) \( f = u + iv \) is real differentiable at \( z_0 \) and

\[
    Df(z_0) = \begin{pmatrix}
        \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0) \\
        \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0)
    \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2,
\]

the Jacobian of \( f \) at \( z_0 \), is \( \mathbb{C} \)-linear.

(iii) \( f = u + iv \) is real differentiable at \( z_0 \) and

\[
    \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0).
\]

(iv) \( f = u + iv \) is real differentiable at \( z_0 \) and \( \frac{\partial f}{\partial \overline{z}}(z_0) = 0 \); in this case \( \frac{\partial f}{\partial \overline{z}}(z_0) = f'(z_0) \). Recall that

\[
    \frac{\partial f}{\partial \overline{z}} := \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} := \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right).
\]

If \( f \) is complex differentiable at each \( z_0 \in \Omega \), then we call \( f \) holomorphic (on \( \Omega \)). The set of all such \( f \) is \( \mathcal{O}(\Omega) \).