COMPLEX ANALYSIS II

Introduction to geometric function theory and Schramm-Loewner evolution

0. Introduction

function theory	$\longleftrightarrow$	geometric function theory
theory of holomorphic functions $f: \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$ open, ...	$\longleftrightarrow$	theory of holomorphic mappings $f: \Omega_1 \to \Omega_2$, $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ open, ...
$\mathcal{O}(\Omega)$	$\longleftrightarrow$	$\mathcal{O}(\Omega_1, \Omega_2)$
<u>typical questions</u>		<u>typical questions</u>
curve integrals, antiderivatives, series expansion, residues, ...		injectivity, surjectivity, bijectivity, $f(\Omega_1)$, f^{-1}, ...

"biholomorphic functions are the isomorphisms of function theory"

$\exists\, f: \Omega_1 \to \Omega_2$ biholomorphic (i.e. f bijective, f, f^{-1} hol.) $\Longrightarrow$ Ω_1 and Ω_2 have "similar" properties in the sense of compl. analysis

Guiding question of geom. function theory

How are the properties of f as a holomorphic function $f : \Omega_1 \to \mathbb{C}$ affected by its properties as a (holomorphic) mapping $f : \Omega_1 \to \Omega_2$?

0.1. Definition:

(i) Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open. A holomorphic function $f : \Omega \to \mathbb{C}$ is called schlicht, if it is injective.

(ii) Consider $\mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \}$. We put

$$\mathcal{S} := \{ f \in \mathcal{O}(\mathbb{D}) \text{ schlicht} \mid f(0) = 0,\ f'(0) = 1 \}.$$

0.2. Theorem (Bieberbach conjecture,
Ludwig Bieberbach, 1916,
Louis de Branges, 1985)

Consider the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

of any $f \in \mathcal{S}$. Then

$$|a_n| \leq n \qquad \forall n \geq 2$$

with equality for any $n \geq 2$ if and only if f is a rotation of the Koebe function.

Our goals:

We want to understand

- some of the main tools of the proof and
- how they are used in modern research.

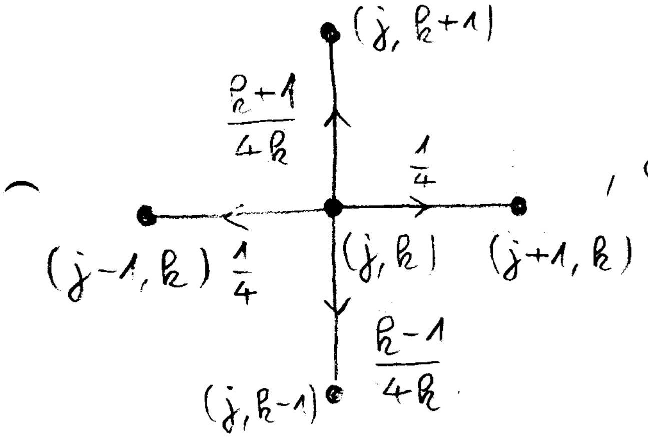
In particular: stochastic Loewner evolution (SLE)

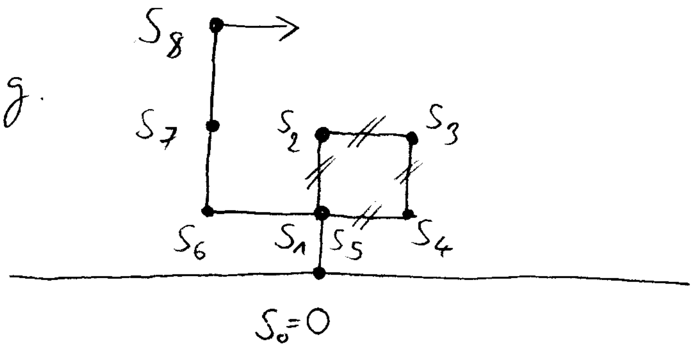
0.3. Example (Loop-erased random walk, LERW)

- Consider the random walk half-plane excursion, i.e., the process $(S_n)_{n=1}^{\infty}$ on

$$\mathbb{Z}_+^2 := \{(j,k) \in \mathbb{Z}^2 \mid k > 0\} \subset \mathbb{C}$$

with transition probabilities given as

- 

, e.g. 

Erase loops in chronological order; obtain the loop-erased random walk $(\hat{S}_n)_{n=0}^{\infty}$; linear interpolation yields $(\hat{S}_t)_{t\geq 0}$; let $(\hat{S}_t^\delta)_{t\geq 0}$ be the scaled process on $\delta\mathbb{Z}_+^2$ for $\delta > 0$.

The scaling limit of $(\hat{S}_t)_{t\geq 0}$ (i.e. $(S_t^\delta)_{t\geq 0}$ for $\delta \downarrow 0$)

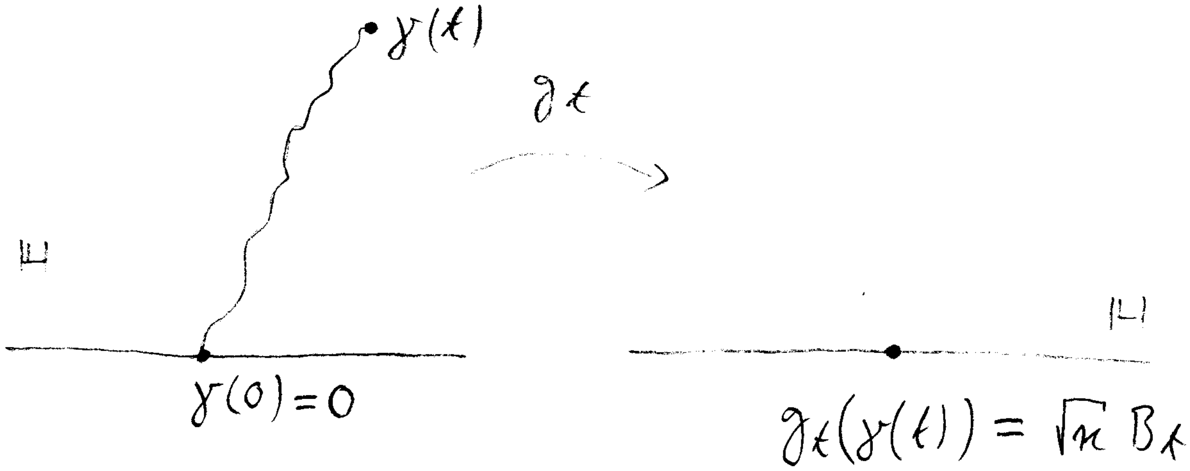
is given by the chordal SLE_2.
(G. Lawler, O. Schramm, W. Werner, 2004)

For any $\kappa > 0$, SLE_κ means the random path γ in $\mathbb{H} := \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$ associated to the random collection $(g_t)_{t \geq 0}$ of biholomorphic maps $g_t : \mathbb{H} \setminus \gamma([0,t]) \to \mathbb{H}$ obtained from

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}\, B_t}, \quad g_0(z) = z$$

one-dimensional Brownian motion

Note that $\gamma : [0, \infty) \to \mathbb{H}$ satisfies $\gamma(0) = 0$ and is injective.



1. Conformal maps

1.1. Reminder:

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open. For $f : \Omega \to \mathbb{C}$ and each $z_0 \in \Omega$, the following statements are equivalent:

(i) f is complex differentiable at z_0, i.e. the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

(ii) $f = u + iv$ is real differentiable at z_0 and

$$Df(z_0) = \begin{pmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0) \\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0) \end{pmatrix} : \underset{\mathbb{C}}{\overset{\mathbb{R}^2}{\shortparallel}} \longrightarrow \underset{\mathbb{C}}{\overset{\mathbb{R}^2}{\shortparallel}},$$

the Jacobian of f at z_0, is $\mathbb{C}$-linear.

(iii) $f = u + iv$ is real differentiable at z_0 and

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0).$$

(iv) $f = u + iv$ is real differentiable at z_0 and $\frac{\partial f}{\partial \bar z}(z_0) = 0$; in this case $\frac{\partial f}{\partial z}(z_0) = f'(z_0)$. Recall that

$$\frac{\partial f}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right).$$

If f is complex differentiable at each $z_0 \in \Omega$, then we call f holomorphic (on Ω). The set of all such f is $\mathcal{O}(\Omega)$.