1.2. Theorem: (Open mapping theorem)

(i) Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be a holomorphic function that is non-constant on each connected component of Ω.
Then $f(\Omega)$ is an open set.

(ii) Let $\emptyset \neq G \subseteq \mathbb{C}$ be a domain and let $f : G \to \mathbb{C}$ be a non-constant holomorphic function.
Then $f(G)$ is a domain as well.

1.3 Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open, $z_0 \in \Omega$, and $f \in \mathcal{O}(\Omega)$.
Then f is injectiv in an open neighborhood of z_0 if and only if $f'(z_0) \neq 0$.

1.4. Theorem

Let $\emptyset \neq \Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open and let $f : \Omega_1 \to \Omega_2$ be holomorphic and bijectiv. Then $f^{-1} : \Omega_2 \to \Omega_1$ is holomorphic as well and we have that

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \qquad \forall w \in \Omega_2$$

In particular, f is biholomorphic. Furthermore, Ω_1 is a simply connected domain if and only if Ω_2 is so.

1.5. Theorem (Riemann mapping theorem)

Let $\emptyset \neq G \subsetneq \mathbb{C}$ be a simply connected domain. Then, for each $z_0 \in G$, there exists a unique biholomorphic mapping $f: G \to \mathbb{D}$ that satisfies

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0.$$

- Biholomorphic mappings enjoy some very important geometric properties. Recall that $\mathbb{C} = \mathbb{R}^2$ is an euclidean space with respect to the inner product

$$\langle \cdot, \cdot \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \quad (z, w) \mapsto \operatorname{Re}(z\overline{w}).$$

1.6. Definition:

- A $\mathbb{R}$-linear, invertible mapping $T: \mathbb{C} \to \mathbb{C}$ is called angle-preserving, if

$$\frac{\langle T(w), T(z) \rangle}{|T(w)| \cdot |T(z)|} = \frac{\langle w, z \rangle}{|w| \cdot |z|} \qquad (1)$$

holds for all $w, z \in \mathbb{C} \setminus \{0\}$, i.e., if

$$\measuredangle(T(w), T(z)) = \measuredangle(w, z) \qquad \forall w, z \in \mathbb{C} \setminus \{0\},$$

where $\measuredangle(w, z) \in [0, \pi]$ denotes the angle between w and z that is determined uniquely by

$$\cos(\sphericalangle(w,z))\,|w|\,|z| = \langle w, z\rangle$$

for all $w, z \in \mathbb{C}\setminus\{0\}$.

1.7. Lemma

Let $T: \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$-linear map.
Then the following statements are equivalent:

(i) T is angle-preserving.

(ii) There exists $a \in \mathbb{C}\setminus\{0\}$, such that either

$$T(z) = az \quad \forall z \in \mathbb{C}$$

or

$$T(z) = a\bar{z} \quad \forall z \in \mathbb{C}.$$

(iii) There is $s > 0$, such that

$$\langle T(w), T(z)\rangle = s\langle w, z\rangle \quad \forall w, z \in \mathbb{C}$$

Proof:

First note that for $w, z \in \mathbb{C}\setminus\{0\}$

$$\langle w, z\rangle = \mathrm{Re}(w\bar{z}) = \mathrm{Re}\left(|z|^2 \frac{w}{z}\right) = |z|^2\,\mathrm{Re}\left(\frac{w}{z}\right),$$

so that $w \perp z$ (i.e., $\sphericalangle(w,z) = \frac{\pi}{2}$) $\Leftrightarrow \frac{w}{z} \in \mathbb{R}i$.

(i) $\Rightarrow$ (ii):

Since $\langle T(i), T(1)\rangle \overset{(1)}{=} 0$ (as $\langle i, 1\rangle = 0$), the above gives

$$\frac{T(i)}{T(1)} \in \mathbb{R}i, \quad \text{say } T(i) = irT(1) \text{ for some } r \in \mathbb{R}$$

Since $\langle T(1+i), T(1-i)\rangle \overset{(1)}{=} 0$ (as $\langle 1+i, 1-i\rangle = 0$),
we get:

$$\begin{aligned} 0 &= \langle T(1+i), T(1-i)\rangle \\ &= \langle T(1)(1+ir), T(1)(1-ir)\rangle \\ &= |T(1)|^2 \operatorname{Re}\left((1+ir)^2\right) \\ &= |T(1)|^2 (1-r^2) \qquad \Longrightarrow \quad r \in \{-1, 1\} \end{aligned}$$

Thus:
$$\begin{aligned} T(z) &= x\,T(1) + y\,T(i) \\ &= \underbrace{T(1)}_{=a}(x \pm iy) \qquad \forall z = x+iy \in \mathbb{C}. \end{aligned}$$

<u>(ii) ⟹ (iii):</u>

Since
- $\langle aw, az\rangle = |a|^2 \langle w, z\rangle \quad \forall w, z \in \mathbb{C}$
- $\langle \bar{w}, \bar{z}\rangle = \langle w, z\rangle \quad \forall w, z \in \mathbb{C}$,

we infer that in both cases

$$\langle T(w), T(z)\rangle = |a|^2 \langle w, z\rangle \quad \forall w, z \in \mathbb{C}.$$

Thus, $s := |a|^2 > 0$ does the job.

<u>(iii) ⟹ (i):</u>

Since $|T(z)| = \sqrt{s}\,|z|$, T is injective and thus invertible. Furthermore, we have that

$$\begin{aligned} |w|\,|z|\,\langle T(w), T(z)\rangle &\overset{(iii)}{=} |w|\,|z|\,s\,\langle w, z\rangle \\ &= |T(w)|\,|T(z)|\,\langle w, z\rangle. \end{aligned}$$

□

1.8 Definition:

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open. We call $f = u + iv \in C^1(\Omega)$

(i) angle-preserving at $z_0 \in \Omega$, if

$$T := Df(z_0) : \mathbb{C} \to \mathbb{C}$$

is angle-preserving in the sense of Definition 1.6.

(ii) orientation-preserving at $z_0 \in \Omega$, if

$$\det(Df(z_0)) > 0$$

(iii) locally conformal, if f is angle-preserving and orientation-preserving at all points $z_0 \in \Omega$

(iv) conformal, if f is locally conformal and injective.

1.9 Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and and consider $f : \Omega \to \mathbb{C}$. Then the following statements are equivalent:

(i) f is locally conformal on Ω

(ii) f is holomorphic on Ω and $f'(z) \neq 0$ for all $z \in \Omega$.

Proof:

(i) $\Rightarrow$ (ii):

Take any $z_0 \in \Omega$.

- f angle-preserving at z_0

  $\Rightarrow$ $Df(z_0) : \mathbb{C} \to \mathbb{C}$ is angle-preserving
- f orientation-preserving at z_0

  $\Rightarrow$ $\det(Df(z_0)) > 0$ (2)

Hence, by Lemma 1.7, there is $a \in \mathbb{C} \setminus \{0\}$, such that

$$Df(z_0)z = az \quad \forall z \in \mathbb{C}.$$

Note that the alternative case

$$Df(z_0)z = a\bar{z} \quad \forall z \in \mathbb{C}$$

would give that

$$Df(z_0)z = (a_1 + ia_2)(x - iy)$$
$$= (a_1x + a_2y) + i(a_2x - a_1y)$$

for all $z = x + iy \in \mathbb{C}$, i.e.

$$Df(z_0) = \begin{pmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{pmatrix},$$

and hence $\det(Df(z_0)) = -|a|^2 < 0$, which contradicts (2).

Thus: $Df(z_0) : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear

$\overset{\text{Reminder 1.1}}{\Longrightarrow}$ f is complex differentiable at z_0

(f being real differentiable at z_0 is ensured by $f \in C^1(\Omega)$)

By Problem 1, Assignment 1A, we thus get

$$|f'(z_0)|^2 = \det(Df(z_0)) \overset{(2)}{>} 0$$

$$\Rightarrow \quad f'(z_0) \neq 0 \qquad \forall z_0 \in \Omega$$

(ii) $\Rightarrow$ (i):

Again by Problem 1, Assignment 1A,

$$\det(Df(z_0)) = |f'(z_0)|^2 > 0,$$

i.e. f is orientation-preserving at each $z_0 \in \Omega$.

Moreover, we have that

$$Df(z_0)z = f'(z_0)z \qquad \forall z \in \mathbb{C},$$

which by Lemma 1.7 implies that f is also angle-preserving at each point $z_0 \in \Omega$.

$\square$

1.10. Remark:

Consider a function $f: \Omega \to \mathbb{C}$. Then the following statements are equivalent

(i) f is conformal

(ii) f is biholomorphic as a mapping $f: \Omega \to f(\Omega)$

Indeed:

(i) $\Rightarrow$ (ii): • f locally conformal on Ω

$$\overset{\text{Thm 1.9}}{\Longrightarrow} \quad f \in \mathcal{O}(\Omega)$$

• f injective (in particular, non-constant on all components of Ω)

$$\overset{\text{Thm 1.2}}{\Longrightarrow} \quad f(\Omega) \text{ is open and}$$
$$f: \Omega \to f(\Omega) \text{ is bijective}$$

$$\overset{\text{Thm 1.4}}{\Longrightarrow} \quad f \text{ is biholomorphic}$$

(ii) $\Rightarrow$ (i): • f injective

$$\Longrightarrow \quad f'(z_0) \neq 0 \quad \forall z_0 \in \Omega$$

• $f \in \mathcal{O}(\Omega)$

$$\overset{\text{Thm 1.9}}{\Longrightarrow} \quad f \text{ locally conformal on } \Omega$$

and even conformal.

$\square$