1.2. Theorem: (Open mapping theorem)

(i) Let \( \Phi \subseteq \Omega \subseteq \mathbb{C} \) be open and let \( f : \Omega \to \mathbb{C} \) be a holomorphic function that is non-constant on each connected component of \( \Omega \).
Then \( f(\Omega) \) is an open set.

(ii) Let \( \Phi \subseteq \Omega \subseteq \mathbb{C} \) be a domain and let \( f : \Omega \to \mathbb{C} \) be a non-constant holomorphic function.
Then \( f(\Omega) \) is a domain as well.

1.3. Theorem:

Let \( \Phi \subseteq \Omega \subseteq \mathbb{C} \) be open, \( z_0 \in \Omega \), and \( f \in \mathcal{O}(\Omega) \).
Then \( f \) is injective in an open neighborhood of \( z_0 \) if and only if \( f'(z_0) \neq 0 \).

1.4. Theorem

Let \( \Phi \subseteq \Omega_1, \Omega_2 \subseteq \mathbb{C} \) be open and let \( f : \Omega_1 \to \Omega_2 \) be holomorphic and bijective. Then \( f^{-1} : \Omega_2 \to \Omega_1 \) is holomorphic as well and we have that

\[
(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \quad \forall w \in \Omega_2
\]

In particular, \( f \) is biholomorphic. Furthermore, \( \Omega_1 \)
is a simply connected domain if and only if \( \Omega_2 \) is so.
1.5. Theorem (Riemann mapping theorem)

Let \( \Phi \neq \emptyset \), \( G \subset \subset \Phi \) be a simply connected domain. Then, for each \( z_0 \in G \), there exists a unique biholomorphic mapping \( f : G \to D \) that satisfies

\[ f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0. \]

Biholomorphic mappings enjoy some very important geometric properties. Recall that \( C = \mathbb{R}^2 \) is an euclidean space with respect to the inner product

\[ \langle \cdot, \cdot \rangle : C \times C \to \mathbb{R}, \quad (z, w) \mapsto \text{Re}(z \overline{w}). \]

1.6. Definition:

A \( \mathbb{R} \)-linear, invertible mapping \( T : C \to C \) is called angle-preserving, if

\[ \frac{\langle T(w), T(z) \rangle}{|T(w)| \cdot |T(z)|} = \frac{\langle w, z \rangle}{|w| \cdot |z|} \quad (1) \]

holds for all \( w, z \in C \setminus \{0\} \), i.e., if

\[ \angle(Tw, Tz) = \angle(w, z) \quad \forall w, z \in C \setminus \{0\}, \]

where \( \angle(w, z) \in [0, \pi] \) denotes the angle between \( w \) and \( z \) that is determined uniquely by
\[ \cos (\arg (w, z)) |w| |z| = \langle w, z \rangle \]
for all \( w, z \in \mathbb{C} \setminus \{0\} \).

1.7. Lemma

Let \( T : \mathbb{C} \to \mathbb{C} \) be a \( \mathbb{R} \)-linear map. Then the following statements are equivalent:

(i) \( T \) is angle-preserving.

(ii) There exists an \( a \in \mathbb{C} \setminus \{0\} \) such that either

\[ T(z) = az \quad \forall z \in \mathbb{C} \]

or

\[ T(z) = a \bar{z} \quad \forall z \in \mathbb{C}. \]

(iii) There is \( s > 0 \), such that

\[ \langle T(w), T(z) \rangle = s \langle w, z \rangle \quad \forall w, z \in \mathbb{C} \]

Proof:

First note that for \( w, z \in \mathbb{C} \setminus \{0\} \)

\[ \langle w, z \rangle = \Re (w \bar{z}) = \Re (|z|^2 \frac{w}{z}) = |z|^2 \Re \left( \frac{w}{z} \right), \]

so that \( w \perp z \) (i.e., \( \arg (w, z) = \frac{\pi}{2} \)) \( \iff \) \( \frac{w}{z} \in \mathbb{R} i \).

(i) \( \Rightarrow \) (ii):

Since \( \langle T(i), T(i) \rangle = 0 \) (as \( \langle i, 1 \rangle = 0 \)), the above give \( \frac{T(i)}{T(1)} \in \mathbb{R} i \), say \( T(i) = ir T(1) \) for some \( r \in \mathbb{R} \).
Since \(<T(\lambda + i), T(\lambda - i)> \overset{\text{[u]}}{=} 0 \) (as \(<\lambda + i, \lambda - i> = 0\)), we get:

\[
0 = <T(\lambda + i), T(\lambda - i)>
= <T(\lambda)(\lambda + ir), T(\lambda)(\lambda - ir)>
= |T(\lambda)|^2 \Re((\lambda + ir)^2)
= |T(\lambda)|^2 (\lambda^2 - r^2) \quad \Rightarrow \quad r \in \{-\lambda, \lambda\}
\]

Thus:
\[
T(z) = xT(\lambda) + yT(i)
= \frac{T(\lambda)(x \pm iy)}{a} \quad \forall z = x + iy \in \mathbb{C}.
\]

\((i) \Rightarrow (iii)\):

Since \(<aw, az> = |a|^2 <w, z> \forall w, z \in \mathbb{C}\)
\[
\quad \cdot <\overline{w}, \overline{z}> = <w, z> \quad \forall w, z \in \mathbb{C},
\]
we infer that in both cases,
\[
<T(w), T(z)> = |a|^2 <w, z> \quad \forall w, z \in \mathbb{C}.
\]
Thus, \(s := |a|^2 > 0\) does the job.

\((iii) \Rightarrow (i)\):

Since \(|T(z)| = |a| |z|\), \(T\) is injective and thus invertible. Furthermore, we have that
\[
|w| |z| <T(w), T(z)> \overset{(iii)}{=} |w| |z| s <w, z>
= |T(w)| |T(z)| <w, z>.
\]
\(\square\)
1.8 Definition:

Let \( \Omega \subset \mathbb{C} \) be open. We call \( f = u + iv \in \mathcal{C}^1(\Omega) \)

(i) angle-preserving at \( z_0 \in \Omega \), if

\[
T := Df(z_0) : \mathbb{C} \to \mathbb{C}
\]

is angle-preserving in the sense of Definition 1.6.

(ii) orientation-preserving at \( z_0 \in \Omega \), if

\[
\det (Df(z_0)) > 0
\]

(iii) locally conformal, if \( f \) is angle-preserving and orientation-preserving at all points \( z_0 \in \Omega \)

(iv) conformal, if \( f \) is locally conformal and injective.

1.9 Theorem:

Let \( \Omega \subset \mathbb{C} \) be open and and consider \( f : \Omega \to \mathbb{C} \). Then the following statements are equivalent:

(i) \( f \) is locally conformal on \( \Omega \)

(ii) \( f \) is holomorphic on \( \Omega \) and \( f'(z) \neq 0 \) for all \( z \in \Omega \).
Proof:

(i) \implies (ii):

Take any \( z_0 \in \Omega \).

- \( f \) angle-preserving at \( z_0 \),
  \[ \implies Df(z_0) : C \to C \text{ is angle-preserving} \]

- \( f \) orientation-preserving at \( z_0 \),
  \[ \implies \det(Df(z_0)) > 0 \quad (2) \]

Hence, by Lemma 1.7, there is \( a \in C \setminus \{0\} \), such that

\[ Df(z_0)z = az \quad \forall z \in C. \]

Note that the alternative case

\[ Df(z_0)z = \bar{a}z \quad \forall z \in C \]

would give that

\[ Df(z_0)z = (a_1 + ia_2)(x - iy) \]

\[ = (a_1 x + a_2 y) + i(g_2 x - a_2 y) \]

for all \( z = x + iy \in C \), i.e.

\[ Df(z_0) = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \]

and hence \( \det(Df(z_0)) = -1|a|^2 < 0 \), which

contradicts (2).
Thus: $Df(z_0): \mathbb{C} \to \mathbb{C}$ is $C$-linear

**Reminder 1.1**

$\Rightarrow$ $f$ is complex differentiable at $z_0$

($f$ being real differentiable at $z_0$

is ensured by $f \in C^1(\Omega)$)

By Problem 1, Assignment 1A, we thus get

$$|f'(z_0)|^2 = \det(Df(z_0)) > 0 \tag{2}$$

$\Rightarrow f'(z_0) \neq 0 \quad \forall z_0 \in \Omega$

(3) $\Rightarrow$ (4):

Again by Problem 1, Assignment 1A,

$$\det(Df(z_0)) = |f'(z_0)|^2 > 0,$$

i.e. $f$ is orientation-preserving at each $z_0 \in \Omega$.

Moreover, we have that

$$Df(z_0)z = f'(z_0)z \quad \forall z \in \mathbb{C},$$

which by Lemma 1.7 implies that $f$ is also angle-preserving at each point $z_0 \in \Omega$.  \[\square\]
Consider a function \( f : \Omega \to \mathbb{C} \). Then the following statements are equivalent:

(i) \( f \) is conformal

(ii) \( f \) is bidilatomorphic as a mapping \( f : \Omega \to f(\Omega) \)

Indeed:

\[(i) \Rightarrow (iii): \cdot \ f \text{ locally conformal on } \Omega \]

\[\Rightarrow \quad f \in \mathcal{O}(\Omega) \]

\[\cdot \ f \text{ injective (in particular, non-constant on all components of } \Omega) \]

\[\Rightarrow \quad f(\Omega) \text{ is open and} \]

\[f : \Omega \to f(\Omega) \text{ is bijective} \]

\[\Rightarrow \quad f \text{ is bidilatomorphic} \]

\[(ii) \Rightarrow (i): \cdot \ f \text{ injective} \]

\[\Rightarrow \quad f'(z_0) \neq 0 \quad \forall z_0 \in \Omega \]

\[\cdot \ f \in \mathcal{O}(\Omega) \]

\[\Rightarrow \quad f \text{ locally conformal on } \Omega \]

and even conformal.