# Assignment 1 A

## Problem 1:

$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x$$

$$= u_x^2 + v_x^2 \qquad (\text{since } u_x = v_y,\ u_y = -v_x)$$

$$= |u_x + i v_x|^2$$

$$= \left|\frac{\partial f}{\partial x}\right|^2 \qquad (\text{since } f = u + iv)$$

$$= |f'|^2 \qquad (*)$$

note that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\partial f}{\partial x}(z_0)$$

By the real inverse function theorem, the following statements are equivalent for $z_0 \in \Omega$

(i) $\exists\, U, V$ open, $z_0 \in U \subseteq \Omega$, $f(z_0) \in V$:

$f|_U : U \to V$ is bijective

(ii) $\det(Df(z_0)) \neq 0 \quad (\overset{(*)}{\Longleftrightarrow} f'(z_0) \neq 0)$,

where (i) is (by the open mapping theorem) equivalent to

$\exists\, U$ open, $z_0 \in U \subseteq \Omega$: $f|_U : U \to \mathbb{C}$ injective

Problem 2:

(i) Let $f \in \mathcal{J}$ be given. Since $\operatorname{ord}(f, 0) = 1$, we have that

$$\tilde{f} : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \begin{cases} \frac{f(z^2)}{z^2} & , z \neq 0 \\ 1 & , z = 1 \end{cases}$$

is holomorphic with $\tilde{f}(0) = 1 \neq 0$. Since $f$ has no zeros on $\mathbb{D} \setminus \{0\}$, also $\hat{f}$ has no zeros on $\mathbb{D} \setminus \{0\}$. Thus, $\hat{f}$ has no zeros on $\mathbb{D}$.

$\Longrightarrow \exists L \in \mathcal{O}(\mathbb{D}) : \quad \tilde{f}(z) = \exp(L(z)) \quad \forall z \in \mathbb{D}$

Thus, $\tilde{g} : \mathbb{D} \to \mathbb{C}, \; z \mapsto \exp\left(\frac{1}{2} L(z)\right)$ satisfies

$$\tilde{g}(z)^2 = \exp\left(\frac{1}{2} L(z)\right)^2 = \exp(L(z)) = \tilde{f}(z).$$

Define $g : \mathbb{D} \to \mathbb{C}, \; z \mapsto \pm z \, \tilde{g}(z)$. Then

$$g(z)^2 = z^2 \tilde{g}(z)^2 = z^2 \tilde{f}(z) = f(z^2).$$

Note that $\tilde{g}(0) \in \{-1, 1\}$, since

$$\tilde{g}(0)^2 = \tilde{f}(0) = 1.$$

We choose the sign in $g$ such that

$$g'(0) = \pm \tilde{g}(1) = 1 \quad (\text{note } g'(z) = z\,\tilde{g}'(z) + \tilde{g}(z))$$

Then: $g$ is a square root transform of $f$.

This proves existence.

<u>Uniqueness</u>: $g_1, g_2$ SRT of $f$

$$\Rightarrow \quad g_1(z)^2 = f(z^2) = g_2(z)^2 \qquad (*)$$

Since • $\operatorname{ord}(g_2, 0) = 1 = \operatorname{ord}(g_1, 0)$

• $g_1, g_2$ have no zeros on $\mathbb{D}\setminus\{0\}$ ($f \in \mathcal{J}$!)

$$\Rightarrow \quad \frac{g_1}{g_2} : \mathbb{D}\setminus\{0\} \to \mathbb{C}$$

admits a hol. extension to $\mathbb{D}$ with

$$\left(\frac{g_1}{g_2}\right)(0) = \lim_{z\to 0} \frac{g_1(z)}{g_2(z)} = \lim_{z\to 0} \frac{g_1(z)/z}{g_2(z)/z}$$

$$= \frac{g_1'(0)}{g_2'(0)} = 1 \qquad (**)$$

By $(*)$, $\frac{g_1}{g_2}$ takes values in $\{-1, 1\}$, is thus constant; hence $\frac{g_1}{g_2} \equiv 1$ by $(**)$.

$$\Rightarrow \quad g_1 = g_2$$

(ii) • With $g$, also $z \mapsto -g(-z)$ is a SRT.

$$\overset{(i)}{\Longrightarrow} \quad g(z) = -g(-z) \quad \forall z \in \mathbb{D}$$

i.e. $g$ is odd.

• $g \in \mathcal{J}$ was shown above.