Assignment 1 B

Problem 1

(i) Since $f \in \mathcal{S}$ is injective on $\mathbb{D}$, $f^{-1}(\{0\}) = \{0\}$, thus f has no zeros on $\mathbb{D} \setminus \{0\}$.

$\Rightarrow$ $f \in \mathcal{T}$.

(ii) to show: g is injective (then $g \in \mathcal{S}$)

Let $z_1, z_2 \in \mathbb{D}$ with $g(z_1) = g(z_2)$ be given.

$\Rightarrow$ $f(z_1^2) = g(z_1)^2 = g(z_2)^2 = f(z_2^2)$

$\Rightarrow$ $z_1^2 = z_2^2$, i.e. $z_1 = z_2$ or $z_1 = -z_2$

Assume that $z_1 = -z_2$. Then, since g is odd,

$$g(z_1) = g(-z_2) = -g(z_2) = -g(z_1)$$

$\Rightarrow$ $g(z_1) = 0$

Thus, since $g \in \mathcal{T}$, $z_1 = 0$. Hence $z_2 = -z_1 = 0$.

In any case $z_1 = z_2$.

(iii) We write

- $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_1 = 1$, $a_0 = 0$
- $g(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, $\alpha_1 = 1$, $\alpha_n = 0$ for n even.

Then:

$$g(z)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \alpha_{n-k} \alpha_k \right) z^n$$

$\Big($ n odd: either $n-k$ or k is even $\forall k = 1, \dots, n \Big)$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n} \alpha_{2n-k} \alpha_k \right) z^{2n}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \alpha_{2(n-k)+1} \alpha_{2k-1} \right) z^{2n}$$

$$f(z^2) = \sum_{n=0}^{\infty} a_n z^{2n}$$

$$\Longrightarrow \quad a_n = \sum_{k=1}^{n} \alpha_{2(n-k)+1} \alpha_{2k-1} \quad \forall n \in \mathbb{N}.$$

Problem 2

(i) $k_\theta(0) = 0$

$$k_\theta'(z) = \frac{(1-ze^{i\theta})^2 + z\cdot 2(1-ze^{i\theta})e^{i\theta}}{(1-ze^{i\theta})^4}$$

$$= \frac{(1-ze^{i\theta}) + 2ze^{i\theta}}{(1-ze^{i\theta})^3}$$

$$= \frac{1+ze^{i\theta}}{(1-ze^{i\theta})^3} \Rightarrow k_\theta'(0) = 1.$$

injectivity follows from (iii).

$\Rightarrow k_\theta \in \mathcal{S}$

(ii) $$k_0(z) = \frac{z}{(1-z)^2} = z\cdot\sum_{n=1}^{\infty} n z^{n-1}$$

$$= \sum_{n=1}^{\infty} n z^n = z + \sum_{n=2}^{\infty} n z^n$$

$$e^{-i\theta} k_0(e^{i\theta}z) = \frac{z}{(1-ze^{i\theta})^2} = k_\theta(z)$$

$$\parallel$$

$$z + e^{-i\theta}\sum_{n=2}^{\infty} n(e^{i\theta}z)^n = z + \sum_{n=2}^{\infty} n e^{i(n-1)\theta} z^n$$

(iii) We check that

$$\operatorname{Re}(\varphi_1(z)) = \operatorname{Re}\left(\frac{(1+z)(1-\bar z)}{|1-z|^2}\right) = \operatorname{Re}\left(\frac{1-|z|^2-2i\,\operatorname{Im}(z)}{|1-z|^2}\right)$$

$$= \frac{1-|z|^2}{|1-z|^2} > 0 \quad \forall z \in \mathbb{D}.$$

$$\left[\begin{aligned} w = \frac{1+z}{1-z} &\iff (1-z)w = 1+z \\ &\iff w-1 = z(w+1) \\ &\iff z = \frac{w-1}{w+1} \end{aligned}\right.$$

<u>Def</u>: $\phi : \mathbb{H} \to \mathbb{C}, \quad w \mapsto \dfrac{w-1}{w+1}$

$$1 - |\phi(w)|^2 = 1 - \frac{|w-1|^2}{|w+1|^2} = \frac{|w+1|^2 - |w-1|^2}{|w+1|^2}$$

$$= \frac{(|w|^2 + 2\,\mathrm{Re}(w) + 1) - (|w|^2 - 2\,\mathrm{Re}(w) + 1)}{|w+1|^2}$$

$$= \frac{4\,\mathrm{Re}(w)}{|w+1|^2} > 0$$

<u>Thus</u>: $\psi_1(\mathbb{D}) \subseteq \mathbb{H}, \quad \phi(\mathbb{H}) \subseteq \mathbb{D}$

$\psi_1 \circ \phi = id_{\mathbb{H}}, \quad \phi \circ \psi_1 = id_{\mathbb{D}}$

$\Longrightarrow \quad \psi_1 : \mathbb{D} \to \mathbb{H}$ biholomorphic $(\psi_1^{-1} = \phi)$

$\psi_2 : \mathbb{H} \to \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ biholomorphic

$\Longrightarrow \quad k_0 = \psi_2 \circ \psi_1 : \mathbb{D} \to \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ bihol

$$\left[\begin{aligned} \psi_2(\psi_1(z)) &= \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) \\ &= \frac{1}{4}\,\frac{(1+z)^2 - (1-z)^2}{(1-z)^2} \\ &= \frac{z}{(1-z)^2} = k_0(z) \end{aligned}\right.$$