Problem 1:

(i) \( g \) satisfies \( \{ f(\mathbb{D}) \subseteq g(\mathbb{D}) \} \implies \Phi : \mathbb{D} \to \mathbb{D}, \ z \mapsto g^{-1}(f(z)) \) is well-defined and holomorphic

\[
\begin{align*}
\Phi(0) &= 0 \\
g(0) &= 0, \text{ i.e. } g^{-1}(0) = 0 \implies \Phi(0) = 0
\end{align*}
\]

Thus, by the Schwarz lemma, \( |\Phi'(0)| \leq 1 \).

On the other hand,

\[
\Phi'(z) = (g^{-1})'(f(z)) \cdot f'(z) \quad \text{(chain rule)}
\]

\[
= \frac{1}{g'(g^{-1}(f(z)))} \cdot f'(z) \quad \text{(Theorem 1.4)}
\]

\[
= \frac{f'(z)}{g'(\Phi(z))},
\]

so that

\[
|\Phi'(0)| = \frac{|f'(0)|}{g'(\Phi(0))} = \frac{|f'(0)|}{g'(0)} = \frac{a_1}{c_1}.
\]

In summary, \( \frac{|a_1|}{|c_1|} = |\Phi'(0)| \leq 1 \); hence

\[
|a_1| \leq |c_1|.
\]
(ii) Given \( r \in [0,1), \Theta \in \mathbb{R} \), we may compute:

\[
P(r e^{i \Theta}) = \frac{1}{m} \sum_{k=1}^{m} \sum_{n=1}^{\infty} a_n \left( s^{k,n} r^{-\frac{1}{m}} e^{i \Theta n} \right)^n
\]

\[
= \sum_{n=1}^{\infty} a_n r^{-\frac{n}{m}} e^{i \Theta n} \left[ \frac{1}{m} \sum_{k=1}^{m} s^{k,n} \right] \left\{ \begin{array}{ll}
1 & \text{if } n = p \cdot m \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
= \sum_{p=1}^{\infty} \alpha_{p \cdot m} r^p e^{i p \Theta}
\]

\[
= \sum_{p=1}^{\infty} \alpha_{p \cdot m} (r e^{i \Theta})^p,
\]

i.e.

\[
P(z) = \sum_{p=1}^{\infty} \alpha_{p \cdot m} z^p.
\]

Thus, \( P \) is well-defined and represented on \( D \) by a convergent (!) power series; thus, \( P \) is holomorphic.

\[(\star) \quad \text{if } n = p \cdot m \quad \Rightarrow \quad s^{k,n} = \exp(2\pi i k p) = 1 \quad \forall k
\]

\[
\Rightarrow \frac{1}{m} \sum_{k=1}^{m} s^{k,n} = 1
\]

• Otherwise, \( s^{n} \neq 1 \), we have

\[
= \frac{1}{m} \sum_{k=1}^{m} s^{n_{k,k}} = \frac{1}{m} \sum_{k=0}^{m-1} (s^{k})^k
\]

\[
= \frac{1}{m} \sum_{k=0}^{m-1} (s^{k})^k = 0
\]
Next, note that \( h(1) \leq g(1) \), as \( g(1) \) is convex by assumption. Indeed:

\[
\begin{align*}
C \text{ convex,} \\
z_1, \ldots, z_m &\in C, \\
\lambda_1, \ldots, \lambda_m &\in [0, 1], \\
\lambda_1 + \ldots + \lambda_m &\equiv 1
\end{align*}
\]

\[
\sum_{k=1}^{m} \lambda_k z_k \in C
\]

\((\text{here: } z_k = f(S_k r^m e^{i \theta_m}))
\lambda_k = \frac{1}{m}\)

Thus, by (i),

\[|\alpha_m| = |h'(0)| \leq |g'(0)| = 1 \Rightarrow |\theta_m| = 1 \quad \forall m \in \mathbb{N}
\]

Since \( m \) was arbitrarily chosen, (ii) follows.

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**Problem 2**

(i) The statement (ii) of Problem 1 says \((f = g)\)

\[|\theta_m| = 1 \Rightarrow |\theta_m| = 1 \quad \forall m \in \mathbb{N}
\]

\[g \in \mathcal{F}
\]

(ii) Try:

\[g : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \sum_{n=1}^\infty z^n = \frac{1}{1-z} - 1
\]

(i.e. \( \theta_m = 1 \quad \forall m \in \mathbb{N} \))

Note that \( g(D) \) is convex (and non-planar since

\[g(z) = \frac{1}{1-z} - 1 = \frac{1 - \frac{1}{2}(1-z)}{1-z} - \frac{1}{2} = \frac{1}{2} \left( \frac{1+z}{1-z} - 1 \right)
\]

\[\Rightarrow g(D) = \frac{1}{2} (1-H-1) = \{ z \in \mathbb{C} \mid \text{Re}(z) > -\frac{1}{2} \}.
\]