Assignment 2A

Problem 1:

(i) $\left.\begin{array}{l} g \text{ schlicht} \\ f(\mathbb{D}) \subseteq g(\mathbb{D}) \end{array}\right\} \Longrightarrow$ $\phi : \mathbb{D} \to \mathbb{D},\ z \mapsto g^{-1}(f(z))$ is well-defined and holomorphic

$$\left.\begin{array}{l} f(0) = 0 \\ g(0) = 0, \text{ i.e. } g^{-1}(0) = 0 \end{array}\right\} \Longrightarrow \phi(0) = 0$$

Thus, by the Schwarz Lemma, $|\phi'(0)| \leq 1$.

On the other hand,

$$\begin{aligned} \phi'(z) &= (g^{-1})'(f(z)) \cdot f'(z) && \text{(chain rule)} \\ &= \frac{1}{g'(g^{-1}(f(z)))} \cdot f'(z) && \text{(Theorem 1.4)} \\ &= \frac{f'(z)}{g'(\phi(z))}, \end{aligned}$$

so that $\phi'(0) = \dfrac{f'(0)}{g'(\phi(0))} = \dfrac{f'(0)}{g'(0)} = \dfrac{a_1}{b_1}$.

In summary, $\dfrac{|a_1|}{|b_1|} = |\phi'(0)| \leq 1$; hence

$$|a_1| \leq |b_1|$$

(ii) Given $r \in [0,1)$, $\theta \in \mathbb{R}$, we may compute:

$$h(re^{i\theta}) = \frac{1}{m} \sum_{k=1}^{m} f(\varsigma^k r^{1/m} e^{i\theta/m})$$

$$= \frac{1}{m} \sum_{k=1}^{m} \sum_{n=1}^{\infty} a_n (\varsigma^k r^{1/m} e^{i\theta/m})^n$$

$$= \sum_{n=1}^{\infty} a_n r^{n/m} e^{in\theta/m} \underbrace{\left[\frac{1}{m} \sum_{k=1}^{m} \varsigma^{k\cdot n} \right]}_{\overset{(*)}{=} \begin{cases} 1, & \text{if } n = p\cdot m \\ 0, & \text{otherwise} \end{cases}}$$

$$= \sum_{p=1}^{\infty} a_{p\cdot m} r^p e^{ip\theta}$$

$$= \sum_{p=1}^{\infty} a_{p\cdot m} (re^{i\theta})^p,$$

i.e. $$h(z) = \sum_{p=1}^{\infty} a_{p\cdot m} z^p.$$

Thus, h is well-defined and represented on $\mathbb{D}$ by a convergent (!) power series; thus, h is holomorphic.

$(*)$: • $n = p\cdot m \Rightarrow \varsigma^{k\cdot n} = \exp(2\pi i k p) = 1 \ \forall k$

$\Rightarrow \frac{1}{m} \sum_{k=1}^{m} \varsigma^{k\cdot n} = 1$

• otherwise, $\varsigma^n \neq 1$, we have $\frac{1}{m} \sum_{k=1}^{m} \varsigma^{nk} = \frac{1}{m} \sum_{k=0}^{m-1} (\varsigma^n)^k$

$= \frac{1}{m} \frac{\varsigma^{nm}-1}{\varsigma^n - 1} = 0$

Next, note that $h(\mathbb{D}) \subseteq g(\mathbb{D})$, as $g(\mathbb{D})$ is convex by assumption. Indeed:

$$\left.\begin{array}{l} C \text{ convex}, \\ z_1, \dots, z_m \in C, \\ \lambda_1, \dots, \lambda_m \in [0,1], \\ \lambda_1 + \dots + \lambda_m = 1 \end{array}\right\} \quad \sum_{k=1}^{m} \lambda_k z_k \in C$$

$$\left(\begin{array}{l} \text{here}: z_k = f(\zeta^k r^{1/m} e^{i\theta/m}) \\ \qquad \lambda_k = 1/m \end{array} \right)$$

Thus, by (i),

$$|a_m| = |h'(0)| \leq |g'(0)| = |b_1|$$

Since m was arbitrarily chosen, (ii) follows.

Problem 2

(i) The statement (ii) of Problem 1 says ($f \overset{!}{=} g$)

$$|b_m| \leq |b_1| \underset{\substack{\uparrow \\ g \in S}}{=} 1 \qquad \forall m \in \mathbb{N}$$

(ii) Try: $g : \mathbb{D} \to \mathbb{C}, \; z \mapsto \sum_{n=1}^{\infty} z^n = \frac{1}{1-z} - 1$

(i.e. $b_m \overset{!}{=} 1 \;\; \forall m \in \mathbb{N}$) $\qquad = \frac{z}{1-z}$

Note that $g(\mathbb{D})$ is convex (and g schlicht) since

$$g(z) = \frac{1}{1-z} - 1 = \frac{1 - \frac{1}{2}(1-z)}{1-z} - \frac{1}{2} = \frac{1}{2}\left(\frac{1+z}{1-z} - 1\right)$$

$$\Rightarrow g(\mathbb{D}) = \frac{1}{2}(\mathbb{H} - 1) = \{ z \in \mathbb{C} \mid \operatorname{Re}(z) > -\tfrac{1}{2} \}.$$