Assignment 2B

Problem 1:

Consider the Koebe transform of f (cf. Prop 2.9)

$$g: \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{f(\phi(z)) - f(a)}{(1-|a|^2)\, f'(a)}$$

Then $g \in \mathcal{S}$ (by Theorem 2.7 (i)) and hence

$$\frac{|z|}{(1+|z|)^2} \leq |g(z)| \leq \frac{|z|}{(1-|z|)^2} \quad \forall z \in \mathbb{D}$$

by the Growth Theorem (Theorem 2.11):

$$\Rightarrow \quad \underbrace{\frac{1}{(1+|z|)^2} \leq \left|\frac{g(z)}{z}\right|} \leq \frac{1}{(1-|z|)^2} \quad \forall z \in \mathbb{D}$$

$$\Rightarrow \quad \liminf_{|z|\to 1} \left|\frac{g(z)}{z}\right| \geq \frac{1}{4}$$

Moreover, since

$$\tilde{g}: \mathbb{D} \to \mathbb{C}, \quad z \mapsto \begin{cases} \frac{z}{g(z)}, & z \neq 0 \\ \frac{1}{g'(0)}, & z = 0 \end{cases}$$

is well-defined and holomorphic, the maximum modulus principle yields for all $r \in (0,1)$

$$1 = \frac{1}{g'(0)} = |\tilde{g}(0)| \leq \max_{|z|=r} |\tilde{g}(z)| = \left[\min_{|z|=r} \left|\frac{g(z)}{z}\right|\right]^{-1}$$

$$\Rightarrow \quad \min_{|z|=r} \left| \frac{g(z)}{z} \right| \leq 1 \qquad \forall\, r \in (0,1)$$

$$\Rightarrow \quad \liminf_{|z|\to 1} \left| \frac{g(z)}{z} \right| \leq 1$$

Thus,

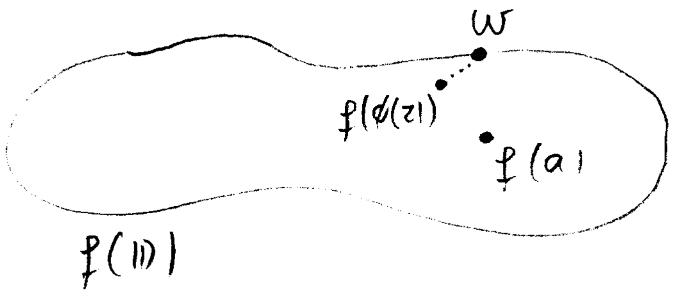
$$\frac{1}{4} \leq \underbrace{\liminf_{|z|\to 1} \left| \frac{g(z)}{z} \right|} \leq 1$$

$$\begin{aligned} &= \liminf_{|z|\to 1} |g(z)| \\ &= \liminf_{|z|\to 1} \frac{|f(\phi(z)) - f(a)|}{(1-|a|^2)\,|f'(a)|} \\ &= \frac{d_f(a)}{(1-|a|^2)\,|f'(a)|}, \end{aligned}$$

from which the assertion follows.

Note that

$$\liminf_{|z|\to 1} |f(\phi(z)) - f(a)| = \inf_{w \in \partial f(\mathbb{D})} |w - f(a)| = d_f(a)$$



Alternative solution for Problem 1

Lemma:
If $f \in \mathcal{S}$, then $\frac{1}{4} \leq d_f(0) \leq 1$

Now, let $f : \mathbb{D} \to \mathbb{C}$ be schlicht and consider

$$g : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{f(\phi(z)) - f(a)}{(1-|a|^2) f'(a)} = \overset{\text{affine linear}}{\lambda}(f(\phi(z)))$$

the Koebe transform of f associated to ϕ. Then

$$d_g(0) = \inf_{w \in \partial g(\mathbb{D})} |w| \qquad \left(\partial g(\mathbb{D}) = \lambda(\partial(f\circ\phi)(\mathbb{D})) \right)$$

$$= \inf_{w \in \underbrace{\partial(f\circ\phi)(\mathbb{D})}_{= \partial f(\mathbb{D}),\ \text{since } \phi(\mathbb{D}) = \mathbb{D}}} \left| \overbrace{\frac{w - f(a)}{(1-|a|^2)|f'(a)|}}^{= \lambda(w)} \right|$$

$$= \frac{1}{(1-|a|^2)|f'(a)|} \underbrace{\inf_{w \in \partial f(\mathbb{D})} |w - f(a)|}_{= d_f(a)}$$

$$= \frac{d_f(a)}{(1-|a|^2)|f'(a)|}$$

From the lemma, the assertion thus follows.

Proof of the lemma:

- $d_f(0) \geq \frac{1}{4}$ follows by the Koebe $\frac{1}{4}$-Theorem.
- $d_f(0) \leq 1$ can be seen as follows:

Assume that $d_f(0) > 1$, i.e.

$$\inf_{w \in \partial f(\mathbb{D})} |w| > 1 + \varepsilon$$

for some $\varepsilon > 0$. Thus

$$D(0, 1+\varepsilon) \subseteq f(\mathbb{D}).$$

Define: $\phi : \mathbb{D} \to \mathbb{D}$, $z \mapsto f^{-1}((1+\varepsilon)z)$

then $\phi(0) = 0$

$\overset{\text{Schwarz}}{\underset{\text{lemma}}{\Longrightarrow}}$ $|\phi'(0)| \leq 1$, where

$$\phi'(0) = (1+\varepsilon)\,(f^{-1})'(0) = (1+\varepsilon)\frac{1}{f'(0)} \overset{f \in \mathcal{S}}{=} 1+\varepsilon,$$

i.e. $|\phi'(0)| > 1$ $\lightning$

Problem 2:

It clearly suffices to prove:

For each $z_0 \in L$, there is an open set $U \subseteq \Omega$ with $z_0 \in \Omega$, such that $\tilde{f}|_U : U \to \mathbb{C}$ is holomorphic.

Given $z_0 \in L$, we choose $r_0 > 0$, such that

$$U := \{ z \in \mathbb{C} \mid |\mathrm{Re}(z - z_0)| < r_0, \underbrace{|\mathrm{Im}(z - z_0)|}_{= \mathrm{Im}(z)} < r_0 \} \subseteq \Omega.$$

Put $U_+ := \Omega_+ \cap U$ and $U_- := \Omega_- \cap U$.

Claim: $\gamma : [a, b] \to U_+ \cup (U \cap \mathbb{R})$ smooth, closed

$$\Rightarrow \int_\gamma f(z)\, dz = 0$$

Proof: For $\varepsilon > 0$, let $U_+^\varepsilon := U_+ - i\varepsilon$ and

$$f_\varepsilon : U_\varepsilon^+ \to \mathbb{C}, \quad z \mapsto f(z + i\varepsilon).$$

Since $\gamma([a,b])$ is compact, there is $\varepsilon_0 > 0$ such that $\gamma([a,b]) \subset U_+^\varepsilon$ for all $0 < \varepsilon < \varepsilon_0$.

Moreover, $\| f - f_\varepsilon \|_\gamma \to 0$ as $\varepsilon \to 0$.

Now,

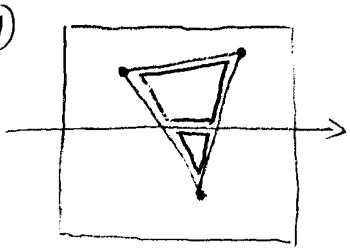
$$\left| \int_\gamma f(z)\, dz - \underbrace{\int_\gamma f_\varepsilon(z)\, dz}_{= 0} \right| \leq L(\gamma)\, \| f - f_\varepsilon \|_\gamma \xrightarrow{\varepsilon \downarrow 0} 0$$

$$\Rightarrow \int_\gamma f(z)\, dz = 0$$

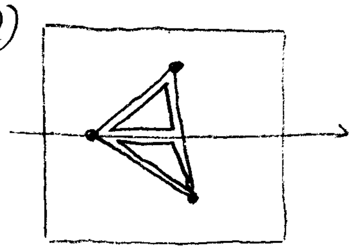
$\square$

Take now $z_1, z_2, z_3 \in U$. Generically (up to symmetry):

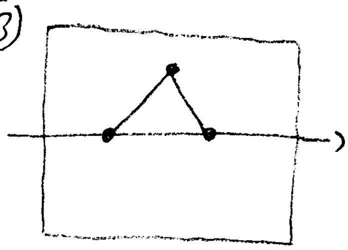
(1)



(2)



(3)



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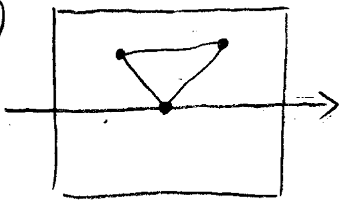

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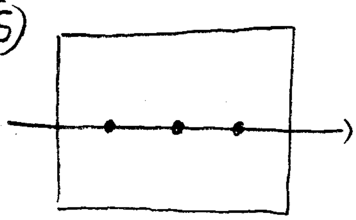

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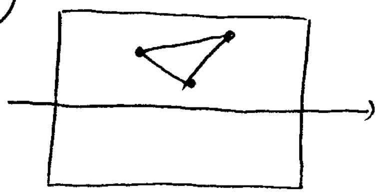

– Due to Morera's theorem, it suffices to prove that

$$\int_{\langle z_1, z_2, z_3 \rangle} \tilde{f}(z)\, dz = 0. \qquad (*)$$

- (1) and (2) can be decomposed as

$$\langle z_1, z_2, z_3 \rangle = \gamma_+ + \gamma_-,$$

with $\gamma_\pm : I_\pm \to U_\pm \cup (U \cap \mathbb{R})$. By the

– claim that was proven above,

$$\int_{\gamma_\pm} \tilde{f}(z)\, dz = 0$$

$\Rightarrow (*)$

- (3) and (4) are treated directly with the claim proved above.
- (5) and (6) are trivial.