Assignment 3A

Problem 1:

Since $(f(D(0,r)))_{r\in(0,1)}$ is an open covering of K (by the Open Mapping Theorem, Theorem 1.2) and increasing, we find $r\in(0,1)$ such that

$$K \subset f(D(0,r)).$$

In particular, $f(\partial D(0,r))\cap K=\emptyset$ and thus

$$\delta := \operatorname{dist}(K, f(\partial D(0,r))) > 0.$$

Since $f_n \to f$ compactly, we find $N\in\mathbb{N}$ such that

$$\|f-f_n\|_{\overline{D(0,r)}} < \delta \quad \text{for all } n\geq N.$$

Now, for all $w\in K$ and $z\in\partial D(0,r)$, we have that

$$\begin{aligned} |(f(z)-w)-(f_n(z)-w)| &= |f(z)-f_n(z)| \\ &\leq \|f-f_n\|_{\overline{D(0,r)}} \\ &< \delta \leq |f(z)-w| \qquad \text{for } n\geq N. \end{aligned}$$

By Rouché, $z\mapsto f(z)-w$ and $z\mapsto f_n(z)-w$ for $n\geq N$ have the same number of zeros on $D(0,r)$ (counted by multiplicity). Since $K\subset f(D(0,r))$, it follows

$$K \subset f_n(D(0,r)) \subseteq f_n(\mathbb{D}) \qquad \text{for all } n\geq N.$$

Problem 2:

"$\Longrightarrow$": Let $f_n \to 0$ compactly as $n \to \infty$.

Assume that a subsequence of $(G_n)_{n=1}^\infty$ has a kernel; w.l.o.g, $(G_n)_{n=1}^\infty$ has a kernel, i.e.

$\exists r > 0, N \in \mathbb{N} \ \forall n \geq N: \quad \overline{D(0,r)} \subset G_n$

Define, for $n \geq N$,

$\phi_n : \mathbb{D} \to \mathbb{D}, \ z \mapsto f_n^{-1}(rz)$.

Since ϕ_n is holomorphic and satisfies $\phi_n(0) = 0$, the Schwarz lemma tells us that

$$\frac{r}{|f_n'(0)|} = |\phi_n'(0)| \leq 1,$$

i.e. $|f_n'(0)| \geq r$ for all $n \geq N$, in contradiction to $f_n \to 0$ (which, in particular, implies that $f_n'(0) \to 0$).

Thus, $(G_n)_{n=1}^\infty$ cannot have a kernel.

"$\Longleftarrow$": Suppose that no subsequence of $(G_n)_{n=1}^\infty$ has a kernel.

Claim: $f_n'(0) \to 0$ as $n \to \infty$

Otherwise, we could find a subsequence $(f_{n_k})_{k=1}^\infty$

such that

$$f_{n_k}'(0) \geq 2\varepsilon \quad \text{for some} \quad \varepsilon > 0.$$

By Theorem 2.8., $(G_{n_k})_{k=1}^{\infty}$ would have a kernel since

$$\overline{D(0, \tfrac{\varepsilon}{4})} \subseteq D(0, \tfrac{1}{4} f_{n_k}'(0)) \subseteq G_{n_k}$$

for all $k \in \mathbb{N}$.

Now, by Theorem 2.11,

$$|f_n(z)| \leq f_n'(0) \frac{|z|}{(1-|z|)^2} \quad \text{for all } z \in \mathbb{D}, n \in \mathbb{N}.$$

Thus, $f_n \to 0$ compactly as $n \to \infty$.